
SIGILLVM ACADEMIÆ HARVARDIANÆ IN NOV: ANG:
CHRISTO ET ECCLESIÆ

COMMENTATIO

DE SUMMATIONE SERIEI

$$\frac{a}{b.(b+d)} + \frac{a}{(b+2d).(b+3d)} + \frac{a}{(b+4d).(b+5d)} + \ldots$$

AB ILLUSTRI SOCIETATE REGIA HAFNIENSI
IN CERTAMINE LITERARIO

PRAEMIO REGIO ORNATA,

AUCTORE
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Ultra!

VIMARIÆ,
SUMTIBUS NOVI BIBLIOPOLII
VULGO LANDES-INDUSTRIE COMPTOIR DICTI.
1818.

TYPIS HOPFERI DE L'ORME.

VIRO

SAGACISSIMO ATQUE CELEBERRIMO

JO. FRIDERICO PFAFFIO,

PROFESSORI MATHESEOS IN ACADEMIA HALENSI ET REGIS BORUSSIAE CONSILIARIO AULICO MERITISSIMO,

PRAECEPTORI SUO AD CINERES USQUE PIE COLENDO MOX COLLEGAE ET AMICO CONJUNCTISSIMO NUNC DESIDERATISSIMO,

HUNC LIBELLUM SACRUM ESSE VOLUIT

AUCTOR.

CONSPECTUS LIBELLI.

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QUAESTIO

AB ILLUSTRI SOCIETATE REGIA HAFNIENSI IN ANNUM 1813 PROPOSITA:

In solutione problematum physico-mathematicorum interdum occurrit hæc series

$$\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\ldots.$$

vel, si terminis generalioribus hæc series exprimatur,

$$\frac{a}{b.(b+d)}+\frac{a}{(b+2d)(b+3d)}+\frac{a}{(b+4d)(b+5d)}+\ldots.$$

Desideratur invenire formulam summatoriam generalem huius seriei, aut saltem demonstrare, quomodo in cito convergentem transformari possit.

COMMENTATIO.

Thema propositum, quum in duas speciales quæstiones, alteram de summatione seriei supra exhibitæ in infinitum excurrentis, alteram de summatione certi numeri terminorum dispesci queat: utramque speciatim considerabimus.

SECTIO I. *Si series proposita in infinitum excurrit.*

A) Prior methodus.

§. 1.

Series $\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\ldots.$ in inf. $= S$

facile ad hasce duas

$\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\ldots.$ in inf. $= s$

$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\ldots.$ in inf. $= t$

reducitur. Est enim

$$s - t = \frac{3-1}{1 \cdot 3} + \frac{7-5}{5 \cdot 7} + \frac{11-9}{9 \cdot 11} + \ldots\ldots$$

$$= 2\left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 7} + \frac{1}{9 \cdot 11} + \ldots\right)$$

$$= 2\,S$$

$$S = \tfrac{1}{2}\,(s - t).$$

Simili modo series generalior

$$P = \frac{x^3}{1 \cdot 3} + \frac{x^7}{5 \cdot 7} + \frac{x^{11}}{9 \cdot 11} + \ldots$$

in hasce duas

$$p = \frac{x^3}{1} + \frac{x^7}{5} + \frac{x^{11}}{9} + \ldots$$

$$q = \frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \ldots$$

resolvitur, quum sit

$$P = \tfrac{1}{2}\,(p - q).$$

Est autem

$$p = x^2\left(\frac{x}{1} + \frac{x^5}{5} + \frac{x^9}{9} + \ldots\right)$$

$$= x^2 \int (1 + x^4 + x^8 + \ldots)\, dx$$

$$= x^2 \int \frac{dx}{1 - x^4}$$

et $\frac{1}{1-x^4}$ in factores resolvendo

$$p = x^2 \int\left(\frac{1:4}{1-x} + \frac{1:4}{1+x} + \frac{1:2}{1+x^2}\right) dx$$

$$= x^2\left(-\tfrac{1}{4}\log(1-x) + \tfrac{1}{4}\log(1+x) + \tfrac{1}{2}\,\text{Arc. tg. } x + \text{Const.}\right)$$

Quum et $\frac{x}{1} + \frac{x^5}{5} + \frac{x^9}{9} + \ldots$ et $-\tfrac{1}{4}\log(1-x) + \tfrac{1}{4}\log(1+x)$ $+\tfrac{1}{2}$ Arc. tg. x pro $x = 0$, ad nihil reducantur, constans ipsa nihilo æquabitur, et habebimus

$$p = x^2\left(-\tfrac{1}{4}\log(1-x) + \tfrac{1}{4}\log(1+x) + \tfrac{1}{2}\,\text{At. } x\right)$$

Similiter alteram tractando, erit

$$q = \frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \ldots = \int (x^2 + x^6 + x^{10} + \ldots)\, dx$$

$$= \int x^2 (1 + x^4 + x^8 + \ldots)\, dx = \int \frac{x^2\, dx}{1 - x^4}$$

$$= \int \left(\frac{1:4}{1-x} + \frac{1:4}{1+x} - \frac{1:2}{1+x^2}\right) dx$$

$$= -\tfrac{1}{4} \log(1-x) + \tfrac{1}{4} \log(1+x) - \tfrac{1}{2} \text{A. tg. } x + \text{Const.}$$

Constantis valor, faciendo $x = 0$, hic quoque invenitur $= 0$, ut sit $q = -\frac{1}{4} \log(1-x) + \frac{1}{4} \log(1+x) - \frac{1}{2}$ Arc. tg. x.

Hinc habemus $P = \frac{1}{2}(p - q)$

$$= \tfrac{1}{8}(1 - x^2) \log(1-x) + \tfrac{1}{8}(x^2 - 1) \log(1+x) + \tfrac{1}{4}(x^2 + 1) \text{Arc. tg. } x.$$

Valorem seriei S si ex hac æquatione, faciendo $x = 0$, deducere volumus, in eam difficultatem incurrimus, quod tunc primus terminus indefinitus $= 0.\infty$ evadit, neque si eum in formam

$\frac{1}{8}(1 + x) \times \frac{1-x}{(\log(1-x))^{-1}}$, quæ pro $x = 1$, est $= \frac{0}{0}$, differentiando numeratorem et denominatorem, definiri potest. Quoties enim differentiationem illam iteres, toties numeratorem $1 - x$, denominatorem $(\log(1-x))^{-f}$, ubi f numeros integros significat, habebis. Alio autem modo hæc difficultas resolvi potest. Considerando enim formam $(1-x) \log(1-x)$, patet eam, maximum vel minimum valorem unice induere pro $-\log(1-x) - \frac{(1-x)}{1-x} = 0$, i. e. $-1 = +\log(1-x)$, $e^{-1} = 1 - x$, $x = 1 - e^{-1} = 0{,}63\ldots$

Quum autem $(1-x)\log(1-x)$, pro $x = 0$, fit $= 0$ pro $x = 1 - e^{-1}$, sit $= -e^{-1} = -0{,}36\ldots$ facile patet, quia hoc nullo alio casu accidit, formam $(x-1)\log(1-x)$ pro illo valore τῆς x, si a signis abstrahimus, *maximam*, non *minimam* evadere. Augendo itaque x inde a valore $0{,}63$, sive minuendo $1-x$ inde a valore $0{,}36$, formæ illius quantitas absoluta usque decrescet, et quidem usque ad nihil, quod erit limes in decrescendo. Quum enim valorum negativorum quantitatis $1-x$, quippe pro quibus forma nostra $(x-1)\log(1-x)$ ima-

ginaria erit, ratio non habenda sit, omnes valores post maximum sequentes inveniri possunt, si pro $1-x$ dimidia, quarta, octava et deinceps partes ejus valoris sumuntur pro quo $(x-1)\log(1-x)$ valorem maximum induit. Nam cujus seriei

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots\ldots$

notum est terminos minores qualibet quantitate, $< \nu$, assignari posse.

Si itaque primum facimus $1-x=\frac{1}{2}.0{,}36\ldots=0{,}18\ldots$, erit $(x-1)\log(1-x) < 0{,}3115$, i. e $< \frac{32}{36}$ valoris maximi. Sumendo deinde pro $(1-x)$ dimidiam partem hujus ultimi valoris, $(x-1)\log(1-x)$ in majore ratione decrescit. Nam a signis abstrahendo alter factor, $x-1$, eadem ratione decrescit; alter, $\log(1-x)$ in proportione arithmetica, i. e. tardius crescit. Quum autem termini seriei

$\frac{32}{36}, (\frac{32}{36})^2, (\frac{32}{36})^3 \ldots\ldots$ in inf. minores quavis quantitate ν fieri possint, hoc majore ratione in serie celerius decrescente fiet. Hinc patet, minuendo $1-x$ eo modo, quem exhibuimus, $(x-1)\log(1-x)$ minorem quavis quantitate evadere. Quum autem semper positiva sit — quia singuli factores $x-1$ et $\log(1-x)$ negativi sunt —, limes ejus in decrescendo erit $=0$. *) Faciendo itaque $1-x=0$ i. e. $x=1$ primus terminus æquationis pro P nihilo æquabitur. Idem, quum secundo termino accidat, pro $x=1$, erit

$$P = \tfrac{1}{4}.2.\text{ Arc. tg. }1$$
$$= \tfrac{1}{2}.\frac{\pi}{4} = \frac{\pi}{8}$$

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*) Idem hac breviori via inveniri, v. cel. J. F. Pfaffius nuper me docuit: sit $y=z\log z$, $z'=mz$, ergo $y'=mz\log mz=mz\log m+mz\log z$, unde habetur pro $z=0$ (quia $\log m$ non infinite magnum fit),

$y'=mz\log z=my$

Quum autem pro $z=0$, etiam $z'=mz=z$, est quoque $y'=y$; quod cum antecedente jungendo, invenitur esse pro $z=0$, $my=y$ quæ æquatio nisi sit $m=1$, quod in hisce calculis subducendis nusquam posuimus, aliter fieri nequit, nisi pro $y=0$. Unde patet, pro $z=0$ esse $z\log z=0$; sive pro $x=1$, $(1-x)\log(1-x)=0$.

Est itaque

$$\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots \text{ in inf.} = \frac{\pi}{8}{}^{*)} = 0,392699081698724154\ldots$$

§. 2.

Pro generaliori series proposita

$$\frac{a}{b.(b+d)} + \frac{a}{(b+2d)(b+3d)} + \ldots\ldots$$

sufficiet hancce

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots\ldots$$

considerare. Licebit etiam numeros b, d pro rationalibus integris inter se primis habere. Si enim irrationales sunt, pro iis fracti ita prope ad eos accedentes, ut diversitas nullius plane momenti sit, substituere; si fracti, per denominatores multiplicando, eos, qui tunc constantem a augebunt, eliminare; si non inter se primi, factorem communem divisione, quæ constantem a minuet, eiicere poterimus. Sit v. c. series

$$\frac{3}{\sqrt{2}.(\sqrt{2}+\sqrt{3})} + \frac{3}{(\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{3})} + \ldots\ldots$$

cujus summa ad fractiones centies millesimas usque inveniri debeat.

Licebit itaque, absque metu erroris, pro $\sqrt{2}$ | 1,4142136; pro $\sqrt{3}$ 1,7320508 substitui. Hinc, ut fractiones eliminentur, in numeratorem et denominatorem 100000000000000 ducamus, unde erit series

$$300000000000000\left(\frac{1}{14142136(14142136+17320508)}\right.$$

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*) Eulerus in introd. in Analys. Infinitor. *t.* 1. c. 8. §. 140. alio modo docet esse

$$\text{Arc. tg. } x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$$

unde prior summatio item facili modo sequitur. Quum autem generalior summatio ita vix inveniri queat, malui hic etiam eandem methodum adhibere quae ad generaliora nos ducet.

$$+\frac{1}{(14142136+2.17320508)(14142136+3.17320508)}+\ldots\ldots)$$

sive dividendo per 4.4, ad eliminandos factores communes

$$18750000000000\left(\frac{1}{3535534(3535534+4330127)}\right.$$

$$\left.+\frac{1}{(3535534+2.4330127)(3535534+3.4330127)}+\ldots\ldots\right)$$

Simili modo series

$$\frac{2}{5.(5+3\frac{1}{3})}+\frac{2}{(5+2.3\frac{1}{3})(5+3.3\frac{1}{3})}+\ldots\ldots$$

in hasce

$$\frac{18}{15.(25+10)}+\frac{18}{(15+2.10)(15+3.10)}+\ldots\ldots)$$

$$\frac{18}{25}\left(\frac{1}{3.(3+2)}+\frac{1}{(3+2.2)(3+3.2)}+\ldots\right)$$

transformabitur; et in utraque specie sufficiet ultimo loco propositas series, quarum denominatores unitati æquantur, summare; quippe quum in summas ita repertas facile numeri 1875,0000000000, sive $\frac{18}{25}$ duci poterunt. — Quarum transformationum eæ, per quas numeros integros rationabiles obtinemus, necessitatis, reliquæ commoditatis tantum causa adhibentur.

§. 3.

In seriem

$$\frac{1}{b.(b+d)}+\frac{1}{(b+2d)(b+3d)}+\ldots\ldots,$$

in qua b et d sint numeri rationabiles, integri, inter se primi, variabilis x ita introducatur, ut habeamus

$$S=\frac{x^{b+d}}{b.(b+d)}+\frac{x^{b+3d}}{(b+2d)(b+3d)}+\frac{x^{b+d5}}{(b+4d)(b+5d)}+\ldots$$

sint etiam

$$s=\frac{x^{b+d}}{b}+\frac{x^{b+3d}}{b+2d}+\frac{x^{b+5d}}{b+4d}+\ldots\ldots$$

$$t = \frac{x^{b+d}}{b+d} + \frac{x^{b+3d}}{b+3d} + \frac{x^{b+5d}}{b+5d} + \ldots\ldots$$

ut habeatur $S = \frac{s-t}{d}$.

$$\text{Est } s = x^{d}\left(\frac{x^{b}}{b} + \frac{x^{b+2d}}{b+2d} + \frac{x^{b+2d}}{b+4d} + \ldots\right)$$

$$= x^{d}\int(x^{b-1} + x^{b+2d-1} + x^{b+4d-1} + \ldots)\, dx$$

$$= x^{d}\int\frac{x^{b-1}\, dx}{1-x^{2d}}$$

$$t = \int(x^{b+d-1} + x^{b+3d-1} + x^{b+5d-1} + \ldots)\, dx = \int\frac{x^{b+d-1}\, dx}{1-x^{2d}}$$

$$S = \frac{1}{d}\left(x^{d}\int\frac{x^{b-1}\, dx}{1-x^{2d}} - \int\frac{x^{b+d-1}}{1-x^{2d}}\right)$$

In horum integralium evolutione b et d ut numeros positivos considerare licebit. Sunt enim quatuor casus diversi 1) ut habeamus et b et d positivos; 2) utrumque negativum; $b = -\beta$, $d = -\delta$; 3) b negativus $= -\beta$, d positivum $= +\delta$; b positivum $= +\beta$ d negativum $= -\delta$. Secundus casus nullo negotio ad primum reducitur, quum sit

$$-\beta.(-\beta-\delta);\ (-\beta-2\delta)(-\beta-3\delta)\ldots.$$
$$= +\beta.(+\delta),\ (\beta+2\delta)(\beta+3\delta)\ldots.$$

Tertio casu habebimus certum m pro quo quantitas $-\beta+2m\delta$ positiva evadat $= \beta'$. Hinc series inde a termino.

$$\frac{x^{\cdot}}{(-\beta+2m\delta)(-\beta+(2m+1)\delta)} + \frac{x^{\cdot}}{(-\beta+(2m+2)\delta)(-\beta+(2m+3)\delta)} + \ldots$$

$$= \frac{x^{\cdot}}{\beta'(\beta'+\delta)} + \frac{x^{\cdot}}{(\beta'+2\delta)(\beta'+3\delta)} + \ldots$$

ad primum casum pertinet. Termini antecedentes

$$\frac{x^{\cdot}}{-\beta.(-\beta+\delta)} + \frac{x^{\cdot}}{(-\beta+2\delta)(-\beta+3\delta)} + \frac{x^{\cdot}}{(-\beta+(2m-2)\delta)(-\beta+(2m-1)\delta)}$$

vulgari methodo sibi invicem addantur, quod plerumque sine molestia fieri poterit. Quorum autem terminorum si magnus numerus

est, infra docebimus, quomodo, quæ hinc nascatur molestia, minuatur *). —

Quartus tandem casus plane cum tertio convenit, indeque ad primum reducitur. Sive enim habeas

$$\frac{1}{\beta.(\beta-\delta)}+\frac{1}{(\beta-2\delta)(\beta-3\delta)}+\ldots,$$

sive

$$\frac{1}{-\beta.(-\beta+\delta)}\mp\frac{1}{(-\beta+2\delta)(-\beta+3\delta)}+\ldots,$$

quia singuli termini utriusque seriei sibi invicem æquales sunt, plane ad idem redit.

Sufficiet porro eas integralium formas explicare, pro quibus $b-1<2d$. Si enim $b-1\geqq 2d$, assignari poterit numerus m, pro quo $b-2md-1<2d$. Faciendo autem $b-2md=\beta$, series de qua quæritur

$$\frac{x^{\cdot}}{b.(b+d)}+\frac{x^{\cdot}}{(b+2d)(b+3d)}+\ldots..$$

ab hacce

$$\frac{x^{\cdot}}{\beta.(\beta+d)}+\frac{x^{\cdot}}{(\beta+2d)(\beta+3d)}+\ldots..$$

tantum terminis

$$\frac{x^{\cdot}}{\beta.(\beta+d)}+\frac{x^{\cdot}}{(\beta+2d)(\beta+3d)}+\ldots+\frac{x^{\cdot}}{(\beta+(2m-2)d(\beta+(2m-1)d)}$$

$$=\frac{x^{\cdot}}{(b-2md)(b-(m-1)d)}+\frac{x^{\cdot}}{(b-(2m-2)d)(b-(2m-3)d)}+\ldots..$$

$$+\frac{x^{\cdot}}{(b-2d)(b-d)}$$

differt,

*) Si quis formulas propositas pro $b=-\beta$, $d=+\delta$ directe integrare vellet, in integralia incideret, quae terminos, quos in contextu a serie nostra separavimus, comprehenderent. Maluimus autem rem aliter exponere, quia constantes hisce integralibus adiiciendi, quippe quum variabilis x et negativos et positivos valores complecteretur, vix definire poterunt.

differt, qui itaque, si integrationem, ponendo β loco b perfecerimus, tantummodo subtrahendi sunt *).

§. 4.

Integralia ipsa hæc sunt **):

$$\int \frac{x^{b-1}dx}{1-x^{2d}} = \frac{1}{d}\sin\frac{b\pi}{d}.\text{Arc.tg.}\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}} - \frac{1}{2d}\cos\frac{b\pi}{d}.\text{lg.}(1-x.\cos\frac{\pi}{d}+x^2)$$

$$+\frac{1}{d}\sin\frac{2b\pi}{d}.\text{A.tg.}\frac{x\sin\frac{2\pi}{d}}{1-x\cos\frac{2\pi}{d}} - \frac{1}{2d}.\cos\frac{2b\pi}{d}.\text{lg.}(1-2x\cos\frac{2\pi}{d}+x^2)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{1}{d}\sin\frac{(d-1)b\pi}{d}.\text{A.tg.}\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}} - \frac{1}{2d}.\cos\frac{(d-1)b\pi}{d}.\text{lg}(1-2x\cos\frac{(d-1)\pi}{d}$$

$$\ldots\ldots \qquad \ldots +x^2)$$

$$-\frac{1}{2d}.\log(1-x) \mp \frac{1}{2d}.\log(1+x) + \text{Const.}$$

ubi habetur signum superius, $\left(-\frac{1}{2d}.\log(1+x)\right)$ pro b numero pari; signum inferius $\left(+\frac{1}{2d}.\log(1+x)\right)$ pro b numero impari.

$\int \frac{x^{b+d-1}dx}{1-x^{2d}}$, si $b+d-1 < 2d$, ex priori integratione fit, ponendo pro b | $b+d$; sin autem $b+d-1 > 2d$, erit

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*) Integratio formulae ipsius pro $b-1 > 2d$ item ad hosce terminos deduceret, qui tamen facilius statim separantur.

**) Vide Eulerum in Calc. integr. vol. 1. cap. 1. num. 80. cf. ejusd. Calc. differ. 5. cap. 18.

$$\int \frac{x^{b+d-1}dx}{1-x^{2d}} = -\int x^{b-d-1}dx + \int \frac{x^{b-d-1}dx}{1-x^{2d}}$$

$$= -\frac{x^{b-d}}{b-d} + \int \frac{x^{b-d-1}dx}{1-x^{2d}}$$

Quod jam superest integrale, quia secundum §um antecedentem $b-1 < 2d$, ergo magis adhuc $b-d-1 < 2d$, ex formula supra exposita, ponendo $b-d$ loco b assignatur. Hinc est

$$\int \frac{x^{b-d-1}dx}{1-x^{2d}} = \frac{-\left[\frac{1}{0}\right] x^{b-d}}{b-d}$$

$$+\frac{1}{d}\left\{\begin{array}{l}
-\sin\frac{(b\pm d)\pi}{d} . \mathrm{A.tg}\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}} - \frac{1}{2}\cos\frac{(b\pm d)\pi}{d} . \lg\left(1-2x\cos\frac{\pi}{d}+x^2\right) \\
+\sin\frac{2(b\pm d)\pi}{d} . \mathrm{A.tg}\frac{x\sin\frac{2\pi}{d}}{1-x\cos\frac{2\pi}{d}} - \frac{1}{2}\cos\frac{2(b\pm d)\pi}{d} . \lg\left(1-2x\cos\frac{2\pi}{d}+x^2\right) \\
+ \ldots\ldots \qquad - \ldots\ldots \\
+\sin\frac{(d-1)(b\pm d)\pi}{d} . \mathrm{A.tg}\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}} - \frac{1}{2}\cos\frac{(d-1)(b\pm d)\pi}{d} . \lg\left(1-2x\cos\frac{(d-1)\pi}{d}\right. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+x^2\right) \\
-\frac{1}{2}\log(1-x) \mp \frac{1}{2}\log(1+x).
\end{array}\right\}$$

Quum autem sit $\sin\left(\frac{b\pi}{d} \pm \pi\right) = \sin\frac{b\pi}{d} . \cos\pi \pm \cos\frac{b\pi}{d} . \sin\pi$

$$= -\sin\frac{b\pi}{d},$$

$$\sin\left(\frac{2b\pi}{d} \pm 2\pi\right) = \sin\frac{2b\pi}{d} . \cos 2\pi \pm \cos\frac{2b\pi}{d} . \sin 2\pi$$

$$= +\sin\frac{2b\pi}{d}$$

et sic porro;

$$\cos\left(\frac{b\pi}{d}\pm\pi\right)=\cos\frac{b\pi}{d}\cdot\cos\pi\mp\sin\frac{b\pi}{d}\cdot\sin\pi$$

$$=-\cos\frac{b\pi}{d}$$

$\cos\left(\frac{2b\pi}{d}\pm 2\pi\right)=+\cos\frac{2b\pi}{d}$ et sic porro; expressio inventa in hancce faciliorem transibit

$$-\left[\tfrac{1}{0}\right]\frac{x^{b-d}}{b-d}+\frac{1}{d}\left(-\sin\frac{b}{d}\pi\,.\,\mathrm{A.tg}\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}}+\tfrac{1}{2}\cos\frac{b}{d}\pi\,.\,\lg.\left(1-2x\cos\frac{2\pi}{d}+x^2\right)\right.$$

$$+\sin\frac{2b}{d}\pi\,.\,\mathrm{A.tg}\frac{x\sin\frac{2\pi}{d}}{1-x\cos\frac{2\pi}{d}}-\tfrac{1}{2}\cos\frac{2b}{d}\pi\,.\,\lg.\left(1-2x\cos\frac{2\pi}{d}+x^2\right)$$

$$-\ldots\ldots\qquad+\ldots\ldots$$

$$\mp\sin\frac{(d-1)b}{d}\pi\,.\,\mathrm{A.tg}\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}}\pm\tfrac{1}{2}\cos\frac{(d-1)b}{d}\pi\,.\,\lg\left(1-2x\cos\frac{(d-1)\pi}{d}+x^2\right)$$

$$\left.-\tfrac{1}{2}\log(1-x)\mp\tfrac{1}{2}\log(1-x)\right)+\text{Const.}$$

Hic si $b+d-1\geqq 2d$ i.e. $b-d\geqq 1$, sive positivum, primus terminus adest, si $b-d<1$ sive negativum, evanescit; si d numerus par, ideoque $d-1$ impar, in terminis antepenultimis valet signum superius; si $b+d$ sive $b-d$ numerus par, in ultimo termino $\left(\tfrac{1}{2}\log(1-x)\right)$ valet signum superius.

Hinc est

$$
S = \frac{\left[\frac{1}{0}\right] x^{b-d}}{d(b-d)}
$$

$$
+\frac{1}{d^2}\left\{
\begin{aligned}
& x^d \left\{
\begin{aligned}
& \sin\frac{b\pi}{d}.\mathrm{A.tg}\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}} - \tfrac{1}{2}\cos\frac{b\pi}{d}.\cos\left(1-2x\cos\frac{\pi}{d}+x^2\right) \\
& +\sin\frac{2b\pi}{d}.\mathrm{A.tg.}\frac{x\sin\frac{2\pi}{d}}{1-x\cos\frac{2\pi}{d}} - \tfrac{1}{2}\cos\frac{2\pi}{d}.\lg\left(1-2x\cos\frac{2\pi}{d}+x^2\right) \\
& + \cdots\cdots \qquad - \cdots\cdots \\
& +\sin\frac{(d-1)b\pi}{d}.\mathrm{A.tg}\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}} - \tfrac{1}{2}\cos\frac{(d-1)b\pi}{d}.\lg\left(1-2x\cos\frac{(d-1)\pi}{d}\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+x^2\right) \\
& -\tfrac{1}{2}\log(1-x) \mp \tfrac{1}{2}\log(1+x)
\end{aligned}
\right\} \\
& +\sin\frac{b\pi}{d}.\mathrm{A.tg}\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}} - \tfrac{1}{2}\cos\frac{b\pi}{d}.\lg\left(1-2x\cos\frac{\pi}{d}+x^2\right) \\
& +\sin\frac{2b\pi}{d}.\mathrm{A.tg.}\frac{x\sin\frac{2\pi}{d}}{1-x\cos\frac{2\pi}{d}} + \tfrac{1}{2}\cos\frac{2b\pi}{d}.\lg\left(1-2x\cos\frac{2\pi}{d}+x^2\right) \\
& + \cdots\cdots \qquad - \cdots\cdots \\
& +\sin\frac{(d-1)b\pi}{d}.\mathrm{A.tg.}\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}} + \cos\frac{(d-1)b\pi}{d}.\lg\left(1-2x\cos\frac{(d-1)\pi}{d}\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+x^2\right) \\
& +\tfrac{1}{2}\log.(1-x) \pm \tfrac{1}{2}\log.(1+x).
\end{aligned}
\right\}
$$

Termino $\frac{1}{2}\log.(1+x)$ signum superius adiicitur pro $b-d$ pari, inferius pro $b-d$ impari.

Pro $b=0$, sive $d=0$, formula proposita directo adhiberi nequit, quod autem nihil nos moratur, quum pro $d=0$, series in hancce abit $\frac{x^{\cdot}}{b^2}+\frac{x^{\cdot}}{b^2}+\frac{x^{\cdot}}{b^2}+\ldots\ldots$,
quam infinite magnam esse per se patet; pro $b=0$, primum terminum $\frac{x^{\cdot}}{0}$ infinite magnum esse apparet, quare reliqui tantum considerandi veniunt, qui, facto $2d=\beta$, in aliam formam, pro qua b (sive β) et d minime nullæ sunt, transformantur.

Quare nihil reliquum est, nisi ut constans adiicienda difiniatur. Quam ipsam $=0$ esse facile patet. Quum enim b et d positivos fecerimus, series ipsa pro $x=0$, evanescit. Idem accidit singulis expressionis summatoriæ terminis, qui partim A. tg. 0, partim log. 1 partim $x=0$ factores habent. Unde Const. $=0-0=0$.

§. 5.

Ut jam ad seriem nostram ipsam revertamur, pro x unitas restituenda est. Quo facto primum Arc. tg. eliminari possunt. Est enim

$$\sin\varphi = 2\sin\tfrac{1}{2}\varphi.\cos\tfrac{1}{2}\varphi,$$

$$1-\cos\varphi = 1-(\cos\tfrac{1}{2}\varphi)^2+(\sin\tfrac{1}{2}\varphi)^2$$

$$= (\sin\tfrac{1}{2}\varphi)^2$$

$$\frac{\sin\varphi}{1-\cos\varphi} = \frac{2\sin\frac{1}{2}\varphi.\cos\frac{1}{2}\varphi}{2(\sin\frac{1}{2}\varphi)^2} = \cot\tfrac{1}{2}\varphi$$

$= \text{tang}\,\frac{\pi-\varphi}{2}$. Hinc, si pro $\varphi\,|\,\frac{\pi}{d},\ \frac{2\pi}{d},\ldots\ldots\frac{(d-1)\pi}{d}$ ponimus,

$$\text{A.tg}\,\frac{x\sin\frac{\pi}{d}}{1-x\cos\frac{\pi}{d}}\ \ldots\ \text{A.tg}\,\frac{x\sin\frac{(d-1)\pi}{d}}{1-x\cos\frac{(d-1)\pi}{d}},$$ quum sit $x=1$, fiunt

$$\frac{(d-1)\pi}{2d},\ \frac{(d-2)\pi}{2d}\ \ldots\ldots\ \frac{\left[\frac{2}{1}\right]\pi}{2d}.$$

Deinde series ipsa, cujus termini hosce factores habent, nempe
$\frac{(d-1)\pi}{d}.\sin\frac{b\pi}{d}+\frac{(d-3)\pi}{d}.\sin\frac{3b\pi}{d}+\ldots+\frac{[\frac{2}{1}]\pi}{d}.\sin\frac{(d-[\frac{2}{1}])b\pi}{d}$
brevissime exprimi potest. Est enim, significante e basin logarithmorum naturalium,

$$\sin u+\sin 3u+\ldots\ldots+\sin\delta u=$$
$$\frac{1}{2\sqrt{-1}}\left\{\begin{matrix}e^{u\sqrt{-1}}+e^{3u\sqrt{-1}}+\ldots\ldots+e^{\delta u\sqrt{-1}}\\-e^{-u\sqrt{-1}}-e^{-3u\sqrt{-1}}-\ldots\ldots-e^{-\delta u\sqrt{-1}}\end{matrix}\right\}=$$
$$\frac{1}{2\sqrt{-1}}\left\{\frac{e^{u\sqrt{-1}}-e^{(\delta+2)u\sqrt{-1}}}{1-e^{2u\sqrt{-1}}}-\frac{e^{-u\sqrt{-1}}-e^{-(\delta+2)u\sqrt{-1}}}{1-e^{-2u\sqrt{-1}}}\right\}.$$

Hujus formulæ terminos ad eundem denominatorem reducondo, et pro binis sibi invicem respondentibus quantitatibus exponentialibus, trigonometricas ponendo, invenies

$$\frac{2\sin u+\sin\delta u-\sin(\delta+2)u}{2-2\cos 2u}=$$
$$\frac{2\sin u+\sin\delta u-\sin(\delta+2)u}{4\sin u^2}.$$

Similiter est

$$\cos u+\cos 3u+\ldots\ldots+\cos\delta u=$$
$$\frac{1}{2}\left\{\begin{matrix}e^{u\sqrt{-1}}+e^{3u\sqrt{-1}}+\ldots\ldots+e^{\delta u\sqrt{-1}}\\+e^{-u\sqrt{-1}}+e^{-3u\sqrt{-1}}\ldots+e^{-\delta u\sqrt{-1}}\end{matrix}\right\}=$$
$$\frac{1}{2}\left\{\frac{e^{u\sqrt{-1}}-e^{(\delta+2)u\sqrt{-1}}}{1-e^{2u\sqrt{-1}}}+\frac{e^{-u\sqrt{-1}}-e^{-(\delta+2)u\sqrt{-1}}}{1-e^{-2u\sqrt{-1}}}\right\},$$

quæ formulæ similiter atque superior tractata in hancce abit

$$\frac{1}{2}.\frac{\cos\delta u-\cos(\delta+2)u}{1-\cos 2u}=$$
$$\frac{1}{4}.\frac{\cos\delta u-\cos(\delta+2)u}{\sin u^2}.$$

Hinc seriem, et, quæ ei æquiparatur, expressionem differentiando, et utrobique signa invertendo, et, quæ inveniuntur, sub eundem denominatorem reducendo, erit

$$\sin u + 3 \sin 3u + \ldots + \partial \sin \partial u =$$

$$\frac{1}{4 \sin u^3} \left\{ \begin{array}{l} \partial . \sin u . \sin \partial u - (\partial + 2) \sin . u \sin (\partial + 2) . u \\ + 2 \cos u . \cos \partial u - 2 \cos u . \cos (\partial + 2) . u \end{array} \right\}$$

Hæc ut ad seriem a sinubus pendentem in expressione supra relata extantem applicentur, illa in hasce duas dispescatur:

$$\pi \left(\sin \frac{b\pi}{d} + \sin \frac{3b\pi}{d} \ldots + \sin \frac{(d-[\frac{2}{1}])b\pi}{d}\right)$$

$$- \frac{\pi}{d} \left(\sin \frac{b\pi}{d} + 3 \sin \frac{3b\pi}{d} + \ldots + (d - \tfrac{2}{1}) \sin \frac{(d-[\frac{2}{1}])b\pi}{d}\right)$$

quæ, si tantisper, brevitatis causa, pro $b\pi$, $d-[\frac{2}{1}]$ literas u, ∂ servamus, fiunt =

$$\frac{\pi}{4 \sin u^2} . (2 \sin u + \sin \partial u - \sin (\partial + 2) u)$$

$$- \frac{\pi}{4 d \sin u^3} \left\{ \begin{array}{l} \partial \sin u \sin \partial u - (\partial + 2) \sin u . \sin (\partial + 2) u \\ + 2 \cos u . \cos \partial u - 2 \cos u . \cos (\partial + 2) u \end{array} \right\} =$$

$$\frac{\pi}{4 d \sin u^3} \left\{ \begin{array}{l} 2 d \sin u^2 + d \sin u . \sin \partial u - d \sin u . \sin (\partial + 2) u \\ \quad - \partial \sin u . \sin \partial u + (\partial + 2) \sin u . \sin (\partial + 2) u \\ \quad - 2 \cos u . \cos \partial u + 2 \cos u . \cos (\partial + 2) u \end{array} \right\}$$

Substituendo jam pro ∂ ejus valorem $d-2$ sive $d-1$, expressio parenthesibus comprehensa utroque casu per diversas in trigonometricis expressionibus solennes transmutationes, simplicissima evadit. Fit enim 1) pro $\partial = d-2$, (propter $d - \partial = 2$, $d - \partial - 2 = 0$)

$$2 d \sin u^2 + 2 \sin u \sin (d-2) u - 2 \cos u \cos (d-2) u + 2 \cos u \cos du$$

$$= 2 d \sin u^2 - 2 \cos (d-1) u + 2 \cos u . \cos du$$

$$= 2 d \sin u^2 - 2 \sin u . \sin du,$$ indeque tota expressio

$$= \frac{\pi}{2 d \sin u^2} . (d \sin u - \sin du)$$

2) Pro $\partial = d-1$, iidem illi termini sunt

$$2 d \sin u^2 + \sin u . \sin (d-1) u + \sin u \sin (d+1) u$$

$$- 2 \cos u . \cos (d-1) u + 2 \cos u . \cos (d+1) u$$

i. e. (quia $- \cos u . \cos (d-1) u + \sin u \sin (d-1) u = - \cos du$

$\cos u \cos (d+1) u + \sin u \sin (d+1) u = + \cos du$)

$= 2\,d \sin u^2 \cdot \cos u . \cos (d-1)\,u + \cos u . \cos (d+1)\,u$
$= 2\,d \sin u^2 - 2 \cos u . \sin u \sin du$

et tota expressio

$$= \frac{\pi}{2\,d \sin u^2} \cdot (d . \sin u - \cos u . \sin du)$$

Substituendo tandem pro u ejus valorem $\frac{b\pi}{d}$, fit $\sin du = \sin b\pi$, i. e., quia b numerus integer est, $= 0$, eamque ob causam, sive $b = d - 2$ sive $= d - 1$ sit, omnes a sinubus pendentes termini

$$= \frac{\pi}{2 \sin \frac{b\pi}{d}}.$$

Quæ expressio eo casu, quo $d = 1$ fallit. Tunc enim termini a sinubus pendentes ipsi evanescunt, minime autem, quod terminus summatorius continet, sunt $\frac{\pi}{0} = \infty$. Quare, ut ubivis adhiberi possit, pro illis a sinubus pendentibus expressionibus poni debet

$\frac{\pi}{2 \sin \frac{b\pi}{d}}$ vel 0 hocque ita declarari, ut pro $d = 1$ habeatur 0,

ceteris casibus $\frac{\pi}{2 \sin \frac{b\pi}{d}}$.

Porro pro $x = 1$, fit $1 - 2x \cos \phi + x^2 = 2\,(1 - \cos \phi) = 4 \sin \frac{1}{2} \phi^2$, indeque $\log (1 - 2x \cos \frac{\pi}{d} + x^2) \ldots \log (1 - 2x \cos \frac{(d-1)\pi}{d} + x^2) =$

$2 \log (2 \sin \frac{\pi}{2d}) \ldots . \; 2 \log (2 \sin \frac{(d-1)\pi}{2d})$.

Tandem $(x^d - 1) \log (1 - x) = (x^{d-1} + x^{d-2} + \ldots + 1)(x - 1) \log (1 - x)$, quia $(x - 1) \log (1 - x)$ nihilo æquiparatur, *) evanescit.

Hinc

*) vide §. 1. hujus libelli.

Hinc fit

$$1)\ldots\; S=\frac{\genfrac{[}{]}{0pt}{}{1}{0}}{d(b-d)}+\frac{1}{d^2}\left\{\begin{array}{l}\left(\dfrac{\pi}{2\sin\dfrac{b\pi}{d}}\ \text{vel}\ 0\right)\\ -2\cos\dfrac{b\pi}{d}\,.\log\left(2\sin\dfrac{\pi}{2d}\right)\\ -2\cos\dfrac{3b\pi}{d}\,.\log\left(2\sin\dfrac{3\pi}{2d}\right)\\ -\ \ldots\ldots\\ -2\cos\dfrac{(d-\genfrac{[}{]}{0pt}{}{2}{1})\,b\pi}{d}\,\log\left(2\sin\dfrac{(d-\genfrac{[}{]}{0pt}{}{2}{1})\,\pi}{2d}\right)\ldots\\ (\mp\log 2\ \text{vel}\ 0)\end{array}\right\}$$

ubi terminus $\frac{1}{d(b-d)}$ tantum adest, si $b-d\geqq 1$; $\frac{\pi}{2\sin\frac{b\pi}{d}}$ si $d>1$; log 2 deficit si b et $b\pm d$ simul pares sive simul impares numero sunt, i. e. si d est numerus par; negativo signo venit, si b par, $b\pm d$ impar; positivo, si b impar, $b\pm d$ par est. Reliquæ ambiguitates per terminorum progressum ipsum facile tolluntur. Quodsi d est numerus par, hæc expressio, quia primus et ultimus, secundus et penultimus, et sic porro, terminorum a cosinibus pendentium, iungi possunt, iterum simplicior evadit,

$$2)\ldots\; S=\frac{\genfrac{[}{]}{0pt}{}{1}{0}}{d(b-d)}+\frac{1}{d^2}\left\{\begin{array}{l}\dfrac{\pi}{2\sin\dfrac{b\pi}{d}}-2\cos\dfrac{b\pi}{d}\,.\log\operatorname{tg}\dfrac{\pi}{2d}\\ -2\cos\dfrac{3b\pi}{d}\,.\log\operatorname{tg}\dfrac{3\pi}{2d}\\ -\ \ldots\ldots\\ -2\cos\dfrac{(\frac{2}{1}d-\genfrac{[}{]}{0pt}{}{2}{1})\,b\pi}{d}\,.\log\operatorname{tg}\dfrac{(\frac{1}{2}d-\genfrac{[}{]}{0pt}{}{2}{1})\,.\,\pi}{2d}\end{array}\right\}$$

in qua ad primam ambignitatem resolvendam regula prima supra posita adhibenda est, reliquæ per progressum ipsum tolluntur.

Hæc formula e superiori ita fit. Pro d numero pari, evanescit ultimus terminus ($\log 2$); quia autem numerus par semper > 1, primus eorum, qui intra parentheses habensur, $\left(\frac{\pi}{2 \sin \frac{b\pi}{d}}\right)$, semper adest; eorum, qui a cosinibus pendent, ultimus fit

$2 \cos \frac{(d-1)b\pi}{d} . \log 2 \sin \frac{(d-1)\pi}{2d}$. Est autem

$\cos \frac{(d-1)b\pi}{d} = \cos b\pi . \cos \frac{b\pi}{d} + \sin b\pi . \sin \frac{b\pi}{d}$, (i. e., quia propter §. 2., ob d parem, b impar est,)

$= - \cos \frac{b\pi}{d}$; $\sin \frac{(d-1)\pi}{2d} = \sin \frac{\pi}{2} . \cos \frac{\pi}{2d} - \cos \frac{\pi}{2} . \sin \frac{\pi}{2d} = \cos \frac{\pi}{2d}$; ergo hic ultimus a cosinibus pendentium terminorum cum primo junctus $= 2 \cos \frac{b\pi}{d}, \log \frac{2 \sin \frac{\pi}{2d}}{2 \cos \frac{\pi}{2d}} = 2 \cos \frac{b\pi}{d} . \log \operatorname{tg} \frac{\pi}{2d}$

Similes ob rationes fiunt secundus et penultimus horum terminorum $= 2 \cos \frac{3b\pi}{d} . \log \operatorname{tg} \frac{3\pi}{2d}$ et sic porro.

Ultimum par terminorum sic jungendorum est

$2 \cos \frac{(\frac{1}{2}d - [\frac{2}{1}])b\pi}{d} \log \operatorname{tg} \frac{(\frac{1}{2}d - [\frac{2}{1}])\pi}{2d}$. Post quod, (pro d pariter pari, ubi 2, non 1 habetur) singulus terminus sequitur, nulli jungendus,

$$2 \cos \frac{\frac{1}{2}db\pi}{d} . \log 2 \sin \frac{\frac{1}{2}d\pi}{2d},$$

qui autem, quia b impar est, propter $\cos \frac{b\pi}{2} = 0$, semper evanescit.

Hinc formula supra posita efficitur *).

*) Formulæ ab ea, quam illustri societati. Hafniensi proposui, eo nomine differunt, quod ibi termini a sinubus pendentes

$$\frac{(d-1)\pi}{d} \cdot \sin\frac{b\pi}{d} + \frac{(d-3)\pi}{d} \cdot \sin 3\frac{b\pi}{d} + \ldots$$

nondum summati, neque pro d numero pari ultimo loco posita summatio adhibita erat. Has quoque summationes adhiberi posse, monuit celeberrimus mihique conjunctissimus D. Prof. de Bohnenberger.

Aliam quoque eamque faciliorem methodum me docuit idem vir cel. formulam summatoriam supra positam inveniendi quam brevissimis hic indicare juvabit. Simplicissima primum seriei forma

$$\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \ldots \text{ in inf.} = P$$

comparari potest cum hacce

$$\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots = \int\frac{dx}{1+x^2} = \text{Arc. tang } x$$

Quia enim hæc series est

$$= \frac{3x - x^3}{1\cdot 3} + \frac{7x^5 - 5x^7}{5\cdot 7} + \ldots\ldots$$

substituendo loco x | 1, habetur

$$P = \tfrac{1}{2} \text{ Arc. tang } 1 = \frac{\pi}{8}$$

Generalior deinde forma

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots\ldots \text{ in inf.} = S$$

simili modo comparatur cum hacce serie

$$\frac{x^b}{b} - \frac{x^{b+d}}{b+d} + \frac{x^{b+2d}}{b+2d} - \frac{x^{b+3d}}{b+3d} + \ldots\ldots$$

$$= \int\frac{x^{b-1}dx}{1+x^d}$$, quae pro $d > b-1$ est

§. 6.

Quæ usquedum exposuimus, ut exemplis illustrentur, consideremus primum series, in quibus b et d jam ita comparati sunt, ut §. 2. 3. supposuimus.

Sit series 1) $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots\ldots$

ubi $b = 1$, $d = 2$. Est itaque, ob d parem, formula 2) adhibenda, ubi, quia $b - d < 1$, primus terminus deficit; eorum, qui a cosinibus et logarithmis tangentium pendent, nullus adest, quia ultimus, propter

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$$= \frac{1}{d}\left\{\begin{array}{l} \sin\frac{b\pi}{d}.\mathrm{A.tg.}\dfrac{x-\cos\frac{\pi}{d}}{\sin\frac{\pi}{d}} - \cos\frac{b\pi}{d}.\log\left(1-2x\cos\frac{\pi}{d}+x^2\right) \\ + \sin\frac{3\pi b}{d}.\mathrm{A.tg}\dfrac{x-\cos\frac{3\pi}{d}}{\sin\frac{3\pi}{d}} - \frac{1}{2}\cos\frac{3b\pi}{d}.\log\left(1-2x\cos\frac{3\pi}{d}+x^2\right) \\ + \ldots\ldots \\ + \sin\frac{(d-[\frac{2}{1}])b\pi}{d}.\mathrm{A.tg}\dfrac{x-\cos\frac{(d-[\frac{2}{1}])\pi}{d}}{\sin\frac{(d-[\frac{2}{1}])\pi}{d}} \\ \qquad - \frac{1}{2}\cos\frac{(d-[\frac{1}{2}])b\pi}{d}.\log\left(1-2x\cos\frac{(d-[\frac{2}{1}]\pi)}{d}+x^2\right) \\ \left|\mp\frac{1}{2}\log(1+x)\ \text{vel}\ 0\right|. \end{array}\right\}$$

Series sic summata, quum sit

$$= \frac{(b+d)x^b - b.x^{b+d}}{b.(b+d)} + \frac{(b+3d)x^{b+2d} - (b+2d)x^{b+3d}}{(b+2d)(b+3d)} + \ldots$$

pro $x = 1$, fit

$$d\left(\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots\right) = d.S,$$

Hinc, in expressione summatoria, pro x ponendo 1, et in reliquis ut supra factum est, procedendo, eadem pro serie nostra summatio habebitur.

$\frac{1}{2}d - [\frac{2}{1}] < 1$, haberi nequit, ut sit

$$S = \frac{1}{4} \cdot \frac{\pi}{2 \sin \frac{\pi}{2}} \text{ sive, quia } \sin \frac{\pi}{2} = 1,$$

$$S = \frac{\pi}{8} \text{ quod idem §. prima invenimus.}$$

Habeamus deinde 2) seriem

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} \cdot \cdot \cdot \cdot \cdot$$

ubi $b = 1$, $d = 1$; ob d imparem formula (1) adhibenda est, in qua primus terminus itemque primus intra parenthesin evanescunt; postremus, quia b impar, $b + d$ par, fit $+ \log 2$; intermedii, quippe quorum ultimus, quia $d - 1 = 0$, non adest, plane deficiunt *). Hinc est $S = \frac{1}{1} \log 2 = 0{,}693147180559945309 \ldots$

Sit porro 3) series

$$\frac{1}{5 \cdot 8} + \frac{1}{11 \cdot 14} + \frac{1}{17 \cdot 20} + \cdot \cdot \cdot \cdot \cdot$$

ubi $b = 5$, $d = 3$, et iterum ex formula (1) computari debet, in qua quia $b - d > 1$, primus terminus adest $= \frac{1}{3 \cdot 2}$. Quia b impar, $b \cdot d$ par, habetur etiam terminus ultimus $= + \log 2$.

Hinc fit

$$S = \frac{1}{3 \cdot 2} + \frac{1}{9} \left(\frac{\pi}{2 \sin \frac{5\pi}{3}} - 2 \cos \frac{5\pi}{3} \cdot \log \left(2 \sin \frac{\pi}{6}\right) + \log 2 \right)$$

$$= \frac{1}{6} + \frac{1}{9} \left(- \frac{\pi}{2 \sin \frac{\pi}{3}} - 2 \cos \frac{\pi}{3} \log \left(2 \sin \frac{\pi}{6}\right) + \log 2 \right)$$

---

*) Revertendo ad eam expressionem in qua x adest, invenimus solum, de quo sermo esse posset, terminum $\log (1 - 2x \cos \frac{\pi}{1} + x^2)$ in logarithmos $(1 + x)$ jam transisse.

et, quia $2\sin\frac{\pi}{6} = 2\sin 30^\circ = 2.\frac{1}{2} = 1$, $\log 1 = 0$,

$$S = \frac{1}{6} + \frac{1}{9}\left(\log 2 - \frac{\pi}{2\sin 60^\circ}\right)$$ sive, quia $\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{3}{2\sqrt{3}}$

$$\frac{\pi}{2\sin 60^\circ} = \frac{\sqrt{3}.\pi}{3}$$

$$\sqrt{3} = 1,7320508046\,80$$

$$\frac{\pi}{3} = 1,0471975511\,97$$

$$\frac{\pi}{2\sin 60^\circ} = 1,8137993612$$

$$\log 2 = 0,6931471806$$

$$S = 0,0421497477.$$

Habeatur 4) ne exemplum complicatioris comput desideretur, series

$$\frac{1}{16.25} + \frac{1}{34.43} + \frac{1}{52.61} + \ldots\ldots,$$

ubi $b = 16$, $d = 9$ et formula (1) adhiberi debet; in qua quia $b - d > 1$, primus terminus adest $= \frac{1}{9.7}$. Quia b par, $b - d$ impar, habetur etiam terminus ultimus, at signo negativo $= -\log 2$. Hinc est

$$S = \frac{1}{7.9}$$

$$+ \frac{1}{9^2}\left\{\begin{matrix} \frac{\pi}{2\sin\frac{16\pi}{9}} - 2\cos\frac{16\pi}{9}.\log\left(2\sin\frac{\pi}{18}\right) - 2\cos.\frac{3.16\pi}{9}.\log\left(2\sin\frac{3\pi}{18}\right) \\ -2\cos\frac{5.16\pi}{9}.\log\left(2\sin\frac{5\pi}{18}\right) - 2\cos\frac{7.16\pi}{9}.\log\left(2\sin\frac{7\pi}{18}\right) - \log 2 \end{matrix}\right\}$$

$$= \frac{1}{7.9} + \frac{1}{9.9}\left\{\begin{matrix} -\frac{\pi}{2\sin\frac{2\pi}{9}} - 2\cos\frac{2\pi}{9}.\log\left(2\sin\frac{\pi}{18}\right) + 2\cos\frac{3\pi}{9}.\log\left(2\sin\frac{3\pi}{18}\right) \\ +2\cos\frac{\pi}{9}.\log\left(2\sin\frac{5\pi}{18}\right) - 2\cos\frac{4\pi}{9}.\log 2\sin\frac{7\pi}{18}\right) - \log 2 \end{matrix}\right\}$$

$$= \frac{1}{7 \cdot 9} + \frac{1}{9 \cdot 9} \left\{ \begin{array}{l} - \frac{90^\circ}{\sin 40^\circ} \\ - 2 \cos 40^\circ . \log (2 \sin 10^\circ) + 2 \cos 60^\circ . \log (2 \sin 30^\circ) \\ + 2 \cos 20^\circ . \log (2 \sin 50^\circ) - 2 \cos 80^\circ . \log (2 \sin 70^\circ) - \log 2 \end{array} \right\}$$

Hinc valor seriei ita accurate computetur ut ope tabularum vulgarium, v. c. Schulzianarum adhibitis portionibus proportionalibus, licet. Habemus $90^\circ = 1{,}570796$

$$\begin{array}{rl} \log 1{,}5.. = & 0{,}1961198 \\ \log \sin 40^\circ = & 9{,}8080675 \\ \hline & 0{,}3880523 \text{, ut sit} \end{array}$$

$$- \frac{90^\circ}{\sin 40^\circ} = - 2{,}443723$$

$$\begin{array}{rl} \log hyp \sin 10^\circ = & - 1{,}75072399 \\ \log hyp\ 2 = & + 0{,}69314718 \\ \hline & - 1{,}05757681 \end{array}$$

$$\begin{array}{rl} \log 1{,}0575.. = & 0{,}0243120 \\ \log \cos 40^\circ = & 9{,}8842540 \\ \log 2 = & 0{,}3010300 \\ \hline & 0{,}2095960 \end{array}$$

et hinc, quum log hyp negativus sit, est $- 2 \cos 40^\circ . \log (2 \sin 10^\circ) = + 1{,}6203020$. Terminus sequens, quia $2 \sin 30^\circ = 1$, indeque $\log hyp\ 2 \sin 30^\circ = 0$, deficit.

$$\begin{array}{rl} \log hyp \sin 50^\circ = & - 0{,}26651509 \\ \log hyp\ 2 = & + 0{,}69314718 \\ \hline & + 0{,}42663209 \end{array}$$

$$\begin{array}{rl} \log 0{,}42663.. = & 9{,}6300535 \\ \log \cos 20^\circ = & 9{,}9729858 \\ \log 2 = & 0{,}3010300 \\ \hline & 9{,}9040693 \end{array}$$

Est itaque $+ 2\cos 20^\circ . \log(2\sin 50^\circ) = + 0{,}8018057$

$$\begin{aligned} \log \text{hyp} \sin 70^\circ &= - 0{,}06220246 \\ \log \text{hyp}\ 2 &= + 0{,}69314718 \\ \hline & \quad 0{,}63094472 \end{aligned}$$

$$\begin{aligned} \log 0{,}63904.. &= 9{,}7999914 \\ \log \cos 80^\circ &= 9{,}2396702 \\ \log 2 &= 0{,}3010300 \\ \hline & \quad 9{,}3406916 \end{aligned}$$

$-2\cos 80^\circ . \log(2\sin 70^\circ) = - 0{,}2191249.$

Est tandem $\log 2 = - 0{,}69314718.$

Hinc habemus terminos positivos

$$\begin{array}{r} 1{,}620302 \\ 0{,}801806 \\ \hline 2{,}422108 \end{array}$$

terminos negativos

$$\begin{array}{r} - 2{,}443723 \\ - 0{,}219125 \\ - 0{,}693147 \\ \hline - 3{,}355995 \end{array}$$

indeque summam omnium in parenthesi comprehensarum quantitatum

$$- 0{,}933887$$

ejusque nonam partem $= - 0{,}1037652$

cui addendo $\frac{1}{7} = + 0{,}1428571$

$$+ 0{,}0390919$$

hanc que summam per 9 dividendo habemus $S = 0{,}00434354$

Facile autem, si ad genesin hujus numeri recurrimus, perspicitur, sex tantum priores figuras decimales certas esse posse, ut certo sit $S = 0{,}004343$.

Sit tandem 5) series $\frac{1}{9 . 19} + \frac{1}{29 . 39} + \ldots .$, ubi $b = 9$, $d = 10$, et formula (2) adhiberi potest. Quia $b - d < 1$, primus terminus deest.

deest. Est itaque

$$S=\frac{1}{100}\left(\frac{\pi}{2\sin\frac{9\pi}{c}}-2\cos\frac{9\pi}{10}.\log tg\frac{\pi}{20}-2\cos\frac{27\pi}{10}.\log tg\frac{3\pi}{20}\right)$$

$$=\left(\frac{90^\circ}{\sin 18^\circ}+2\cos 18^\circ.\log tg\, 9^\circ+2\cos 54^\circ.\log tg.\, 27^\circ\right)$$

Habemus $90^\circ = 1{,}570796$, $\log 1{,}5.. = 0{,}1951198$.
$\log\sin 18^\circ = 9{,}4899824$

0,7051374

ergo $\frac{90^\circ}{\sin 18^\circ}$ = 5,07151

log hyp tg. $9^\circ = -1{,}842730$, $\log 1{,}8.. = 0{,}2654616$
$\log\cos 18^\circ = 9{,}9782063$
$\log 2 = 0{,}3010300$.

0,5446979

itaque $2\cos 18^\circ.\log tg\, 9^\circ = -3{,}50508$

log hyp tg $27^\circ = -0{,}67427549$, $\log 0{,}67.. = 9{,}8288373$
$\log\cos 54^\circ = 9{,}7692187$
$\log 2 = 0{,}3010300$

9,8990860

itaque $2\cos 54^\circ \log tg\, 27^\circ = -0{,}79266$

0,77377

$S = 0{,}0077377$.

§. 7.

Serierum, quæ, antequam summentur, transformari debent, exempla hæc sunto. Habeatur

1) series $S=\frac{2}{11\frac{2}{3}.15}+\frac{2}{18\frac{1}{3}.21}+\ldots\ldots$

ubi d est numerus fractus. Eliminetur itaque primum fractio, quod fit multiplicando per 9, et habebitur

$$\frac{18}{35\cdot45}+\frac{18}{55\cdot65}+\ldots\ldots$$

Quum jam $b=35$, $d=10$ inter se non sint numeri primi, dividatur per factoris communis quadratum, ut habeamus

$$\frac{18}{25}\left(\frac{1}{7\cdot9}+\frac{1}{11\cdot13}+\ldots\ldots\right)$$

Series nunc consideranda

$$\frac{1}{7\cdot9}+\frac{1}{11\cdot13}+\ldots\ldots$$

in eo tantum ab iis diversa est, quas summavimus, quod $b-1>2d$. Est enim $b-1=6$, $2d=4$. Quia autem,

series $\frac{1}{3\cdot5}+\frac{1}{7\cdot9}+\frac{1}{11\cdot13}\ldots\ldots$ ubi $b-1=2, 2d=4$ ergo $b-1<2d$ ex antea expositis summari potest, et est

$$S=\frac{18}{25}\left(\frac{1}{3\cdot5}+\frac{1}{7\cdot9}+\frac{1}{11\cdot23}+\ldots\ldots-\frac{1}{3\cdot5}\right)\,^{*)},$$

tota summatio ex prius deductis perfici potest. In serie

$$\frac{1}{3\cdot5}+\frac{1}{7\cdot9}+\frac{1}{11\cdot13}+\ldots\ldots$$

est $b=3$, $d=2$, ut formula (2) adhiberi possit. Quia $b-d=1$, adest primus terminus $=\frac{1}{2\cdot1}$; termini autem a cosinibus pendentes, quia ultimus non datur, omnes deficiunt. Hinc habetur series

$$=\frac{1}{2}+\frac{1}{4}\cdot\frac{\pi}{2\sin\frac{3\pi}{2}}\text{ sive (quia }\sin\frac{3\pi}{2}=-\sin\frac{\pi}{2}=-1\text{)}$$

$$=\frac{1}{2}-\frac{\pi}{8}=0{,}1073009183012275845\ldots$$

---

*) vide hujus libelli §. 3. in fine.

Hinc subtra hendo $\frac{1}{15} = 0{,}0666666666666666666\ldots$

$$0{,}0406342516346609179\ldots$$

et differentiam, quia $\frac{18}{25} = \frac{8.9}{100}$, multiplicando per 8. 9, dividendo per 100

$$0{,}3250740130768734 32$$
$$2{,}9256661176918660888$$

est $S = 0{,}029256661176918 60888$

Habeatur 2) series

$$S = \frac{1}{11.9} + \frac{1}{7.5} + \ldots\ldots$$

quæ forma seriei ab illustri societate propositæ pro $b = +11$, $d = -2$, sive $b = -11$, $d = +2$ subest. Ex antea observatis *) facile deducitur, seriem si a quarto termino initium ejus fit, talem evadere, in qua et b et d positivi sunt. Hisce enim factis, fit

$\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots\ldots$, quam §. 1. invenimus esse $= \frac{\pi}{8}$. Cui addendo terminos tres priores $\frac{1}{99} + \frac{1}{35} + \frac{1}{3}$, est

$$S = 0{,}76470371083592 4616\ldots\ldots$$

§. 8.

Antequam ad alteram summationis methodum progrediamur, sequentes de priori observationes adiicere juvabit.

Observatio I.) Sunt casus, iique haud pauci, quibus methodo jam exposita summatio nullo fere negotio, ope tabularum pro circumferentia circuli, logarithmorum naturalium, functionúmque trigonometricarum, exactius perfici potest, quam umquam opus erit, quod v. c. in ea serie, quam illustris societas ipsa numeris proposuit, et in reliquis, quas supra computavimus, excepto tantummodo quarto exemplo §. 6tæ, accidit. Sunt autem alii, quibus eadem methodus longe

*) vide hujus libelli §. 3.

difficilior evadit, iique duplicis generis, primum, si differentia d magnus numerus, præcipue impar, est; deinde, si seriei primus terminus ita comparatus est, ut ei parti, quæ summari potest, termini addendi sive inde subtrahendi sint. Prioris difficultatis exemplum §. 6ta num. 4. habuimus; posterioris exempla series

$$\frac{1}{399\cdot 397}+\frac{1}{395\cdot 393}\ldots\ldots \text{ in inf.}$$

vel $\frac{1}{399\cdot 401}+\frac{1}{403\cdot 405}\ldots\ldots$ in inf.

quarum prior est

$$\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\ldots \text{ in inf.}+\frac{1}{399\cdot 397}+\frac{1}{395\cdot 393}+\ldots\ldots\frac{1}{3\cdot 1}$$

posterior

$$\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\ldots \text{ in inf.}-\left(\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{395\cdot 397}\right)$$

ita ut in utraque, præter facilem seriei infinitæ summationem, fere ducenti termini iique ipsi computatu haud faciles, addendi vel subtrahendi veniant. Operæ itaque pretium erit, diligenter inquirere, quomodo hæ difficultates tolli possint. Quod, quoad posteriorem, ad summationem seriei nostræ finitæ, v. c.

$$\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\ldots\ldots+\frac{1}{397\cdot 399}$$

$$\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{395\cdot 397}$$

deducere facile perspicitur; priori autem alia methodus series nostras infinitas summandi medebitur.

Observatio II. Via, qua usquedum ingressi sumus, præter series ab illustri societate propositas, alias etiam facile summari posse, hic breviter indicandum est. Seriebus enim, quas usquedum consideravimus, numeratores adiici possunt, ut sit

$$S=\frac{x^{b+d}}{b.(b+d)}+\frac{x^{b+3d}}{(b+2d)(b+3d)}+\frac{x^{b+5d}}{(b+4d)(b+5d)}+\ldots$$

sive, separando x^{b+d}

$$T = \frac{S}{x^{b+d}} = \frac{1}{b \cdot b+d} + \frac{x^{2d}}{(b+2d)(b+3d)} + \frac{x^{4d}}{(b+4d)(b+5d)} + \ldots$$

sive, ponendo y pro $\frac{1}{x^{2d}}$,

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} \cdot \frac{1}{y} + \frac{1}{(b+4d)(b+5d)} \cdot \frac{1}{y^2} + \ldots\ldots$$ *)

eaque summatio semper procedit, si $x < 1$ sive $y > 1$. Formula summatoria pro S habetur §. 4. unde patet pro $x < 1$, omnes terminos possibiles evadere, quod pro $x > 1$ secus fit **). Hinc est, quia $y = x^{-2d}$, $x = y^{-\frac{1}{2d}}$, $x^{b+d} = y^{-\frac{b+d}{2d}}$ $T = S . y^{\frac{b+d}{2d}}$

Sit v. c. proposita series $\frac{1}{1.3} + \frac{1}{5.7} \cdot \frac{1}{2} + \frac{9}{9.11} \cdot \frac{1}{2^2} + \frac{1}{13.15} \cdot \frac{1}{2^3} + \ldots \pm T$ ubi $b=1$, $d=2$, $y=2$, ergo $x = 2^{-\frac{1}{4}}$, $T = 2^{\frac{3}{4}} S$, S, quia, ob $b - d < 1$ primus terminus deficit, ob b et $b+d$ impar, signum inferius valet, erit

$$= \frac{1}{2} \left\{ \begin{array}{l} x^2 \left(\sin\frac{\pi}{2} . \mathrm{A.tg} \dfrac{x \sin\frac{\pi}{2}}{1 - x\cos\frac{\pi}{2}} - \frac{1}{2}\cos\frac{\pi}{2} \log(\ldots) - \frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x)\right) \\ + \sin\frac{\pi}{2} . \mathrm{A.tg} \dfrac{x \sin\frac{\pi}{2}}{1 - x\cos\frac{\pi}{2}} - \frac{1}{2}\cos\frac{\pi}{2} . \log(\ldots) + \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) \end{array} \right\}$$

*) Haec series ope formulae generalis Bohnenbergerianae, unde vir celeberrimus nostram deduxit, summari nequit: omnino autem aliae eaeque complicatiores.

**) Quum enim tangentes nullos limites habeant, ab Arc. tangentium nihil impossibile oriri potest. ($\mathrm{Log.}(1 - 2x\cos\varphi + x^2)$, quum $1 - 2x\cos\varphi + x^2$ nunquam negativum fit, item impossibiles fieri nequeunt. Sed $\log(1-x)$ impossibilis fit pro $x > 1$. Anne hoc impossibile eliminari possit, nunc disquirere non vacat.

et, quia $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $x = 2^{-\frac{1}{2}}$

$$= \frac{1}{4}\left\{ \begin{array}{l} 2^{-\frac{1}{2}}(\text{A.tg.}\,2^{-\frac{1}{4}} - \frac{1}{2}\log(1-2^{-\frac{1}{4}}) + \frac{1}{2}\log(1+2^{-\frac{1}{4}})) \\ + \text{A.tg.}\,2^{-\frac{1}{4}} + \frac{1}{2}\log(1-2^{-\frac{1}{4}}) + \frac{1}{2}\log(1+2^{-\frac{1}{4}}) \end{array} \right\}$$

$$= \frac{1}{4}(1+2^{-\frac{1}{2}})\,\text{A.tg.}\,2^{-\frac{1}{4}} - \frac{1}{8}(1-2^{-\frac{1}{2}})\log\left(\frac{1+2^{-\frac{1}{4}}}{1-2^{-\frac{1}{4}}}\right)$$

sive, quia $\log 2^{-\frac{1}{4}} = 9{,}9247425$

et hinc $\text{A.tg.}\,2^{-\frac{1}{4}} = 40^\circ\ 3'\ 37'',3$

$= 0{,}69918520$

$1+2^{-\frac{1}{4}} = 1{,}8408964$

$1-2^{-\frac{1}{4}} = 0{,}1591036$

$\log\text{nat}\,(1+2^{-\frac{1}{4}}) = 0{,}6102516$

$\log\text{nat}\,(1-2^{-\frac{1}{4}}) = -1{,}8381998$

$\log\text{nat}\,\left(\frac{1+2^{-\frac{1}{4}}}{1-2^{-\frac{1}{4}}}\right) = 2{,}4484514$

$1+2^{-\frac{1}{2}} = 1{,}7071068$

$\frac{1}{4}(1+2^{-\frac{1}{2}}) = 0{,}4267767$

$(1-2^{-\frac{1}{2}}) = 0{,}2928932$

$\frac{1}{8}(1-2^{-\frac{1}{2}}) = 0{,}03661165$

$S = 0{,}4267767 \cdot 0{,}69918520 - 0{,}03661165 \cdot 2{,}4484514$

$= 0{,}2983959 - 0{,}08964184$

$= 0{,}2087541$

$T = 2^{\frac{1}{4}} \cdot S = 0{,}3510813.$

Habemus itaque

$$\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7}\cdot\frac{1}{2} + \frac{1}{9\cdot 11}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{13\cdot 15}\cdot\left(\frac{1}{2}\right)^3 + \ldots$$

$= 0{,}351081$

Quum sinus et cosinus arcuum multiplorum, ope quantitatum imaginariarum ad potestates reduci queant, hac eadem methodo etiam series hujus formæ

$$\frac{\sin(b+d)\varphi}{b.(b+d)} + \frac{\sin(b+3d)\varphi}{(b+2d)(b+3d)} + \frac{\sin(b+5d)\varphi}{(b+4d)(b+5d)} + \ldots\ldots$$

sive

$$\frac{\cos(b+d)\varphi}{b.(b+d)} + \frac{\cos(b+3d)\varphi}{(b+2d)(b+3d)} + \frac{\cos(b+5d)\varphi}{(b+4d)(b+5d)} + \ldots\ldots$$

summari posse, vix dubitari potest. Quod autem diligantius exponere neque hujus loci est, neque nunc vacat.

B. *Altera methodus.*

§. 9.

Haud ignotæ sunt hæ summationes

$$\frac{1}{(p+1)(p+2)} + \frac{1}{(p+2)(p+3)} + \ldots + \frac{1}{(p+x)(p+x+1)} = \frac{1}{p+1} - \frac{1}{p+x+1}$$

$$\frac{1}{(p+1)(p+2)(p+3)} + \frac{1}{(p+2)(p+3)(p+4)} + \ldots + \frac{1}{(p+x)(p+x+1)(p+x+2)} =$$
$$\frac{1}{2}\cdot\frac{1}{(p+1)(p+2)} - \frac{1}{2}\cdot\frac{1}{(p+x+1)(p+x+2)}$$

et generatim

$$\frac{1}{(p+1)..(p+f)} + \frac{1}{(p+2)..(p+f+1)} + \ldots + \frac{1}{(p+x)..(p+x+f-1)} =$$
$$\frac{1}{f-1}\cdot\frac{1}{(p+1)\ldots(p+f-1)} - \frac{1}{f-1}\cdot\frac{1}{(p+x+1)\ldots(p+x+f-1)}$$

Hinc fit, si facimus $p = \frac{b}{2d} - 1$, dividendo per $(2d)^2$, $(2d)^3\ldots$ $(2d)^f$,

$$\frac{1}{b.(b+2d)} + \frac{1}{(b+2d)(b+4d)} + \ldots + \frac{1}{(b+2(x-1)d)(b+2xd)} = \frac{1}{2d}\left(\frac{1}{b} - \frac{1}{b+2xd}\right)$$
$$= s'$$

$$\frac{1}{b.(b+2d)(b+4d)}+\frac{1}{(b+2d)(b+4d)(b+6d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+2xd)(b+2(x+1)d)}$$

$$=\frac{1}{2.2d}\left(\frac{1}{b.b+2xd}-\frac{1}{(b+2xd)(b+2(x+1)d)}\right)=s''$$

$$\frac{1}{b.(b+2d)..(b+2(f-1)d)}+\frac{1}{(b+2d)..(b+2fd)}+\ldots+\frac{1}{(b+2(x-1)d)\ldots(b+2(x+f-2)d)}$$

$$=\frac{1}{(f-1)2d}\left(\frac{1}{b(b+2d)..(b+2(f-2)d)}-\frac{1}{(b+2xd)..(b+2(x+f-2)d)}\right)$$
$$=s^{(f-1)}$$

Series nostra

$$\frac{1}{b.(b+d)}+\frac{1}{(b+2d)(b+3d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+(2x-1)d)}=S$$

cum prima harum serierum

$$\frac{1}{b.(b+2d)}+\frac{1}{(b+2d)(b+4d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+2xd)}=s'$$

facile comparari potest. Est enim, singulos a singulis terminis subtrahendo

$$S-s'=+d.\left(\frac{1}{b.(b+d)(b+2d)}+\frac{1}{(b+2d)(b+3d)(b+4d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+(2x-1)d)(b+2xd)}\right)$$

Hæc series eodem modo facile cum secunda s'' comparatur, ut sit

$$S-s'-ds''=d.3d.\left(\frac{1}{b.(b+d)(b+2d)(b+4d)}+\frac{1}{(b+2d)(b+3d)(b+4d)(b+6d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+(2x-1)d)(b+2xd)(b+2(x+1)d)}\right)$$

Quam seriem cum tertia superiorum comparando, fit

$S-s'$

$$S - s' - d.s'' - 1.3\,d^2.s''' = 1.3.5\,d^3\left(\frac{1}{b.(b+d)(b+2d)(b+4d)(b+6d)} + \ldots\right.$$

$$\left. + \frac{1}{b+2(x-1)d)(b+(2x-1)d)(b+2x\,d)(b+2(x+1)d)(b+2(x+4d)}\right)$$

et, dum eodem modo continuamus, generatim

$$S - s' - 1.ds'' - 1.3\,d^2.s''' - \ldots\ldots - 1.3.5\ldots(2f-3)\,d^{f-1}s^{(f)} =$$

$$1.3.5\ldots(2f-1)\,d^f.\left(\frac{1}{b.(b+d)(b+2d)(b+4d)\ldots(b+2fd)} + \ldots\right.$$

$$\left. + \frac{1}{(b+2(x-1)d)(b+(2x-1)d)(b+2xd)\ldots(b+2(x+f-1)d)}\right)$$

Habetur itaque, si novas sic ortas series $\Delta', \Delta'', \Delta''' \ldots \Delta^{(f)}$ vocamus,

$$S = s' + \Delta'$$
$$= s' + 1.\,d.s'' + \Delta''$$
$$= s' + 1.\,d.s' + 1.3.\,d^2.\,s''' + \Delta'''$$
$$\ldots\ldots = s' + 1.\,d.s' + 1.3\,d^2.\,s''' + \ldots + 1.3.5\ldots(2f-3)\,d^{f-1}.\,s^{(f)} + \Delta^{(f)}$$

Harum expressionum termini ultimi, $\Delta', \Delta'' \ldots \Delta^{(f)}$, si $b > d$, decrescunt, ut, pro diversa serierum considerandarum indole, vel citius vel tardius, at semper aliquando, tuto negligi possint. Quod ut demonstretur, faciamus b et d esse positivos simulque $b > d$, quod per transformationes faciles obtineri posse §. 3. monstravimus *). Quibus factis sit series nostra, demto primo termino,

$S - \frac{1}{b.(b+d)} = T$, ut habeatur

$$T = \left(\frac{1}{(b+2d)(b+3d)} + \frac{1}{(b+4d)(b+5d)} + \ldots + \frac{1}{(b+2(x-1)d)(b+(x2-1)d)}\right)$$

Sit porro prima serierum, cum quibus eam comparavimus, demto ultimo termino, $s' - \frac{1}{(b+2(x-1)d)(b+2xd)} = t$, ut habeatur

$$t = \frac{-1}{b.(b+2d)} + \frac{1}{(b+2d)(b+4d)} + \ldots\ldots + \frac{1}{(b+2(x-2)d)(b+2(x-1)d)}$$

*) Pro b negativo, d positivo, expressionem saepius divergentem esse experimenta docebunt.

Quarum expressionum terminos singulos cum singulis comparando, patet, quia $b < b+2d$, $b+2d < b+3d$ et sic porro, esse

$$t > T, \text{ i. e. } s' - \frac{1}{(b+2(x-1)d)(b+2xd)} > S - \frac{1}{b.(b+d)}$$

(sive, quia $S = s' + \Delta'$,) $> s' + \Delta' - \frac{1}{b.(b+d)}$

Unde habetur

$$\Delta' < \frac{1}{b.(b+d)} - \frac{1}{(b+2(x-1)d)(b+2xd)}$$

Similiter in reliquis progredi licet. Sit enim generatim series, quæ in $\Delta^{(f-1)}$ comprehensa est, demto primo termino, $= U$, ut habeatur

$$U = \frac{\Delta^{(f-1)}}{1.3\ldots(2f-3)d^{f-1}} - \frac{1}{b.(b+d)(b+2d)\ldots(b+2(f-1)d)}$$

$$= \frac{1}{(b+2d)(b+3d)(b+4d)\ldots(b+2fd)} + \frac{1}{(b+4d)(b+5d)(b+6d)\ldots(b+2(f+1)d)}$$

$$+ \ldots + \frac{1}{(b+2(x-1)d)(b+(2x-1)d)(b+2xd)\ldots(b+2(x+f-2)d)}$$

Sit porro $s^{(f)}$, demto ultimo termino, $= u$, ut habeatur

$$u = s^{(f)} - \frac{1}{(b+2(x-1)d)\ldots(b+2(x+f-1)d)}$$

$$u = \frac{1}{b.(b+2d)\ldots(b+2fd)} + \frac{1}{(b+2d)(b+4d)\ldots(b+2(f+1)d)} + \ldots$$

$$+ \frac{1}{(b+2(x-1)d)(b+2xd)\ldots(b+2(x+f-2)d)}$$

Jam, quum $b < b+2d$, $b+2d < b+3d$; $b+4d$, $b+6d\ldots b+2fd$ in utraque serie inveniantur, primus terminus seriei u major est primo termino seriei U; idemque ex similibus argumentis in reliquis terminis obtinet. Est itaque

$$u > U$$

id est

$$s^{f} - \frac{1}{(b+2(x-1)d)\ldots(b+2(x+f-1)d)} > \frac{\Delta^{(f-1)}}{1.3\ldots(2f-3)\,d^{f-1}}$$
$$- \frac{1}{b.(b+d)(b+2d)\ldots(b+2(f-1)d)}$$

Est autem, quia habetur

$$S = s' + 1.\,d.\,s'' + \ldots + 1.3\ldots(2f-5)\,d^{f-2}.\,s^{(f-1)} + \Delta^{(f-1)}$$
$$= s' + \ldots + 1.3\ldots(2f-5)\,d^{f-2}.\,s^{f-1} + 1.3\ldots(2f-3).\,d^{f-1}.\,s^{f}.+\Delta^{(f)}$$
$$\Delta^{(f-1)} = 1.3\ldots(2f-3)\,d^{f-1}.\,s^{(f)} + \Delta^{(f)}$$

sive

$$\frac{\Delta^{(f-1)}}{1.3\ldots(2f-3).\,d^{f-1}} = s^{f} + \frac{\Delta^{(f)}}{1.3\ldots(2f-3).\,d^{f-1}}$$

et hinc

$$s^{f} - \frac{1}{(b+2(x-1)d)\ldots(b+2(x+f-1)d)} > s^{(f)} + \frac{\Delta^{(f)}}{1.3\ldots(2f-3).\,d^{f-1}}$$
$$- \frac{1}{b.(b+d)(b+2d)\ldots(b+2(f-1)d)}$$

$$\Delta^{(f)} < \frac{1.3\ldots(2f-3)\,d^{f-1}}{b.(b+d)(b+2d)\ldots(b+2(f-1)d)} - \frac{1.3\ldots(2f-3)\,d^{f-1}}{(b+2(x-1)d)\ldots(b+2(x+f-1)d)}$$

Jam, quum per se pateat quantitatum, quas modo cum differentiis $\Delta', \Delta'' \ldots \Delta^{(f)}$ comparavimus, terminos positivos majores esse negativis, dum demonstraverimus, illos minores quavis quantitate dabili fieri posse, differentias ipsas $< \,$, fieri posse patebit. Sunt autem illi termini pro $f = 1, 2, 3, 4 \ldots f$

$$\frac{1}{b.(b+d)};\ \frac{d}{b}\cdot\frac{1}{(b+2d)(b+2d)};$$
$$\frac{d}{b}\cdot\frac{3d}{(b+2d)}\cdot\frac{1}{(b+d)(b+4d)};\ \frac{d.3d.5d}{b.(b+2d).(b+4d)}\cdot\frac{1}{(b+d).(b+6d)};$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\frac{d}{b}\cdot\frac{3d}{b+2d}\cdot\frac{5d}{b+4d}\cdots\frac{(2f-3)d}{b+2(f-2)d}\cdot\frac{1}{(b+d)(b+2(f-1)d)}$$

Fractiones singulæ $\frac{d}{b}, \frac{3d}{b+2d}, \frac{5d}{b+4d} \ldots \frac{(2f-3)d}{b+2(f-2)d}$ unitate

minores sunt, quia, ut fecimus, $d < b$, et, eam ob causam, terminus positivus, qui differentiæ generali $\Delta^{(f)}$ respondet

$$< \frac{1}{(b+d)(b+2(f-1)d)}$$

quæ quantitas pro infinite crescente f minor quavis quantitate dabili fieri potest.

Quo facto etiam differentia quantitatis inventæ a vero seriei valore minor est quavis quantitate dabili $\Delta^{(f)} < \nu$. Sic itaque demonstratum est, expressionem pro serie nostra

$$S = s' + 1.d.s'' + 1.3.d^2.s''' + 1.3.5.d^3.s'_v + \ldots \text{ in inf.}$$

cujus terminus generalis est

$$1.3.5\ldots(2f-3)\, d^{f-1}.s^{(f)},$$

licet ex infinite multis terminis constantem — semper enim complementum aliquod Δ', Δ''... Δ^f deest — tamen ad proxime inveniendum seriei valorem utiliter adhiberi posse. Semper enim, si expressionis illius certus numerus terminorum computatus est, ex prioribus inveniri potest, intra quosnam limites vitium stetit; et, vice versa, numerus terminorum hujus seriei (f), ad quem usque computatio extendi debet, ut certi simus, ne vitium certos limites transgrediatur, assignari potest *).

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*) Viro sagacissimo de disciplinis mathematicis meritissimo J. F. Pfaffio, cujus institutioni diligentissimæ si quod in hoc genere scientiarum profeci, debeo, primam ideam comparationis seriei nostræ cum iis, quas hac paragrapho consideravimus, acceptam refero. Quum enim a. 1799, dum academiam Helmstadiensem ornaret, analysin infinitorum doceret, demonstraturus seriem

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots.$$

$$= \frac{2}{1\cdot3} + \frac{2}{5\cdot7} + \ldots. \text{ finitam esse, eam cum hacce}$$

$$\frac{2}{1\cdot5} + \frac{2}{5\cdot9} + \ldots.$$ comparavit, quod nescio an alius ante eum fecerit.

§. 10.

Ut jam, quæ paragrapho antecedente generatim exposita sunt, applicationi aptiora fiant, primum ad id attendendum est, pro serie nostra in infinitum excurrente, de qua adhuc solum loquimur, omnes terminos ab x pendentes, quia $x = \infty$, evanescere, unde est

$$s' = \frac{1}{2d} \cdot \frac{1}{b}$$

$$s'' = \frac{1}{2} \cdot \frac{1}{2d} \cdot \frac{1}{b.(b+2d)}$$

$$s''' = \frac{1}{3} \cdot \frac{1}{2d} \cdot \frac{1}{b.(b+2d)(b+4d)}$$

. . . .

$s^{(f)} = \frac{1}{f} \cdot \frac{1}{2d} \cdot \frac{1}{b.(b+2d)\ldots.(b+2(f-1)d)}$ et hinc series nostra ipsa, pro $s', s''\ldots$ hosce valores substituendo, fit

$$S = \begin{cases} \frac{1}{2bd} + \frac{1}{2.2} \cdot \frac{1}{b.(b+2d)} + \frac{1.3.d}{2.3} \cdot \frac{1}{b.(b+2d)(b+4d)} \\ + \frac{1.3.5.d^2}{2.4} \cdot \frac{1}{b.(b+2d)(b+4d)(b+6d)} + \ldots\ldots \\ + \frac{1.3.5\ldots(2f-3)\,d^{f-2}}{2f} \cdot \frac{1}{b.(b+2d)\ldots(b+2(f-1)d)} + \ldots\ldots \end{cases}$$

Quomodo hæc expressio commodissime computetur, ex regulis generalibus facile patet. Logarithmos, nisi exactius, quam ut tabulæ sufficiant, seriem computare velis, commode adhiberi posse, patet. Id etiam commoditatis est, sequentem terminum ab antecedente facili modo dependere, ut fit ftus

$\frac{(f-1)(2f-3)}{f(b+2(f-1).d)}$ proxime antecedentis.

Hinc habetur hæcce expressio recurrens

$$S = \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots. + \mathfrak{F}^{-1} + \mathfrak{F} + \ldots.$$

ubi est $\mathfrak{A} = \frac{1}{2bd}$

$$\mathfrak{B} = \frac{d}{2(b+2d)} \cdot \mathfrak{A}$$

$$\mathfrak{C} = \frac{2.3.d}{3.(b+4d)} \cdot \mathfrak{B}$$

$$\mathfrak{D} = \frac{3.5.d}{4(b+6d)} \cdot \mathfrak{C}$$

. . . .

$$\mathfrak{F} = \frac{(f-1)(2f-3)d}{f(b+2(f-1)d)} \mathfrak{F}^{-1}$$

Si jam hujus expressionis terminos cum iis quantitatibus comparamus, quas §o proxime antecedente cum differentiis $\Delta', \Delta'', \ldots \Delta^{(f)}$ contendimus, patet terminum generalem ftum,

$$t^{(f)} = \frac{1.3.5\ldots(2f-3)\,d^{f-2}}{2f} \cdot \frac{1}{b.(b+2d)\ldots(b+2(f-1)d)}$$

esse $= \frac{b+d}{2f.d}$ illius quantitatis generalis ftæ

$$1.3.5\ldots(2f-3)\,d^{f-1} \cdot \frac{1}{b.(b+d)(b+2d)(b+4d)\ldots(b+2(f-1)d}$$

quæ quantitas, quod supra demonstravimus, est $> \Delta^{(f)}$, unde habetur

$$\Delta^{(f)} < \frac{2d.f}{b+d} \cdot t^{(f)}$$

Sic itaque, expressionis nostræ terminos computando, facillime statim patet, quota figura decimalis certa sit, quota incerta.

Hoc etiam patet, quo majorem esse quantitatis b — quam æque ac d positivam esse jam antea supposuimus — ad quantitatis d rationem eo celerius expressionem nostram convergere. Quare utile esse potest, ne nimis multos expressionis ipsius terminos computare necesse habeamus, seriei propositæ terminos aliquot priores vulgari mothodo computare, quo facto primus denominator primi termini seriei, quæ superest, summandæ (b) majorem, ad differentiam (d) rationem induit. Quod enim in priori methodo necesse habuimus, ut sit $b-1 < 2d$ eo in hac altera minime opus est. Ex com-

paratione

$$\Delta^{(f)} < \frac{2df}{b+d} \cdot t^{(f)}$$

facillime inferri licet, convergentiam expressionis nostræ, quamvis, quod aliunde patet, numquam plane cessat, tamen semper tardiorem fieri. Quare raro magnus terminorum numerus utiliter computatur. Haud absurde potius hanc regulam tibi feceris, ne sit numerus terminorum computandorum

$$f > \frac{b+d}{2d}$$

quia tantum pro

$f \overset{<}{=} \frac{b+d}{2d}$, vitium $\Delta^{(f)} < \frac{t^{(f)}}{2d}$, i. e. minus ea quantitate, quæ ex ultimo termino summationi seriei ipsi addicitur. Quodsi hi termini non sufficiunt, potius est, majorem terminorum seriei numerum modo vulgato computare, quo majus b obtineatur. Quia autem expressionem semel computari cœptam deserere, eique aliam substituere difficultates haud parvas parere solet, utile esse potest, si tibi propositum est certum numerum figurarum decimalium invenire, ante cœptum computum absolutam expressionem pro vitio

$$\Delta^{(f)} < \frac{1.3\ldots(2f-3)\,d^{f-1}}{b.(b+d)(b+2d)\ldots(b+2(f-1)d)}$$

sive faciliorem at minus accuratam

$$\Delta^{(f)} < \frac{1}{(b+d)(b+2(f-1)d)}$$

considerare. Si enim hinc patet, ut vitium sat parvum fiat, esse faciendum $f > \frac{b+d}{2d}$, hoc tibi indicium esse potest, aliquot adhuc terminos seriei modo vulgari computandos esse, ut b crescat. Et hoc modo tam diu per experimenta continuandum est, donec pro $f \overset{=}{<} \frac{b+d}{2d}$, error $\Delta^{(f)}$ sat parvus factus sit.

Præter eas seriei transformationes, quibus opus est, ut b et d numeri positivi fiant, nullæ, ut hæc methodus adhiberi possit, necessariæ sunt; utiles præter eas, quibus ratio quantitatis b ad d magna fiat, *interdum* adhuc eæ, quibus b et d numeri integri inter se primi fiant. Interdum autem, si hæ transformationes ipsæ nimis operosæ sunt, commodius est, summationem ipsam absque illis institui. Sic etiam, ut loco irrationalium rationales numeri proximi ponantur, opus non est.

Quod in priori methodo locum invenit, summationem eo complicatiorem fieri, quo major sit d, id hic etiam aliter est.

Ex quibus omnibus coniici licet in multis casibus, ubi prior methodus propter multas transformationes, vel propter majorem quantitatis d valorem valde operosa est, hancce faciliorem esse.

§. 11.

Quæ omnia, quo clariora fiant, exempla adiicio. 1) Series proposita sit

$$\frac{1}{101.104} + \frac{1}{107.110} + \frac{1}{113.117} + \ldots$$

ubi $b=101$, $d=3$. Hic quia b permagnam ad d rationem habet, absque consideratione expressionis pro $\Delta^{(f)}$ patet, multas figuras decimales sat commode ita computari posse, ut series summanda a termino $\frac{1}{101.104}$ ipso incipiat. In expressione recurrente pro S est

$$\mathfrak{A} = \frac{1}{606}$$

$$\mathfrak{B} = \frac{3}{2.107} \cdot \mathfrak{A}$$

$$\mathfrak{C} = \frac{2.3}{113} \cdot \mathfrak{B}$$

$$\mathfrak{D} = \frac{3.5.3}{4.119} \cdot \mathfrak{C}$$

$\mathfrak{E} =$

$$\mathfrak{E} = \frac{4 \cdot 7 \cdot 3}{5 \cdot 125} \cdot \mathfrak{D}$$

et hinc

$$S = \left\{ \begin{array}{l} 0{,}001650165 \\ +0{,}000023133 \\ +0{,}000001228 \\ +0{,}000000116 \\ +0{,}000000015 \end{array} \right\}$$

qui quinque termini, ut 7 figuræ decimales certæ sint, sufficiunt.

Est enim error

$$\Delta^{(f)} < \frac{5 \cdot 6}{104} \cdot 0{,}000000015$$
$$< 0{,}00000000468.$$

Illos terminos addendo habemus

$$S = 0{,}0016674657.$$

Quibus quinque terminis si quatuor alios adiicimus, quod, quia horum paucas, tantum figuras decimales computare necesse est, sine molestia perfici potest, decem priores figuræ decimales vitio vacuæ erunt. Sic habetur

$$S = \left\{ \begin{array}{l} 0{,}001650165016 \\ +\ 0{,}000023133154 \\ +\ 0{,}000001228306 \\ +\ 0{,}000000116122 \\ +\ 0{,}000000015607 \\ +\ 0{,}000000002681 \\ +\ 0{,}000000000553 \\ +\ 0{,}000000000118 \\ +\ 0{,}000000000029 \\ \hline \end{array} \right\}$$
$$0{,}001674661586$$

Hic decimam figuram ab errore liberam esse inde patet, quia

$$\Delta^{IX} < \frac{6.9.\ 0{,}000000000029}{104}$$
$$< 0{,}00000000000016.$$

Hunc computum longe faciliorem esse eo, quo hæc series secundum priorem methodum summaretur, facile patet. Ibi enim, quum series summanda ab eo termino incipi debeat, pro quo $b - 1 < 2d$, habemus

$$\frac{1}{5.8} + \frac{1}{11.14} + \ldots\ldots \text{ in inf.}$$
$$- \frac{1}{5.8} - \frac{1}{11.14} - \ldots\ldots - \frac{1}{95.98}$$

quarum prior §. 6. num. 3 facile quidem summata est: at operose subtrahendi essent quindecim termini iique non omnes facile computandi, $\frac{1}{5.8} + \frac{1}{11.14} + \ldots + \frac{1}{95.98}$.

2) Series proposita sit

$$\frac{1}{16.25} + \frac{1}{34.43} + \frac{1}{52.61} + \ldots\ldots$$

quam non sine tædio §. 6. num. 4. ita summavimus ut sexta figura decimalis certa esset. Hac etiam methoda adhibita, si a primo termino inde seriem summare institueremus, computus operosior esset, quia $b = 16$ non multo major quam $d = 9$ esset. At modo vulgari computando quatuor primos terminos

$$\frac{1}{16.25} + \frac{1}{34.43} + \frac{1}{52.61} + \frac{1}{70.79}$$

restat series

$$\frac{1}{88.97} + \frac{1}{106.115} + \ldots\ldots$$

pro qua $b = 88$, $d = 9$. Hic, quia $\frac{b+d}{2d} = \frac{97}{18} > 5$, quinque fere termini expressionis commode computantur, pro quibus error

$$\Delta^{v} < \frac{1.3.5.7.9^4}{88.97.106.124.142.160}$$
$$< 0{,}0000005$$

ut sexta figura decimalis certa sit. Si autem septimam invenire propositum est, adhuc terminos modo communi computemus,

$$\frac{1}{88.97} + \frac{1}{116.115}$$

ut sit series summanda

$$\frac{1}{124.133} + \frac{1}{142.151} + \ldots\ldots, \; b = 124, \; d = 9.$$

Jam, quia $\frac{b+d}{2d} = \frac{133}{18} > 7$, septem priores expressionis termini commode computantur, pro quibus est

$$\Delta^{VII} < \frac{1.3.5.7.9.11.9^6}{124.133.142.160.178.196.214.232}$$
$$< 0{,}000000009 \; ^{*)}$$

ut septima figura decimalis certa sit.

Jam est

$$\frac{1}{16.25} = \frac{1}{400} = 0{,}0025$$
$$\frac{1}{34.43} = 0{,}000683995$$
$$\frac{1}{52.61} = 0{,}000315259$$
$$\frac{1}{70.79} = 0{,}000180832$$
$$\frac{1}{88.97} = 0{,}000117151$$
$$\frac{1}{106.115} = 0{,}000082224$$

$$0{,}003879461$$

*) Haec multiplicatio et divisio per logarithmes quorum priores duae vel tres figurae decimales sufficiunt, facillime procedit.

et in expressione summatoria pro serie quæ restat,

$$\mathfrak{A} = \frac{1}{18.124}, \quad \mathfrak{B} = \frac{9}{1.142} . \mathfrak{A}, \quad \mathfrak{C} = \frac{2.9}{160} . \mathfrak{B}$$

$$\mathfrak{D} = \frac{3.5.9}{4.178} . \mathfrak{C}, \quad \mathfrak{E} = \frac{4.7.9}{5.196} . \mathfrak{D}$$

$$\mathfrak{F} = \frac{5.9.9}{6.214} \mathfrak{E}, \quad \mathfrak{G} = \frac{6.11.9}{7.232} . \mathfrak{F},$$ unde est

$$S = \left\{ \begin{array}{r} 0{,}000448029 \\ +\ 0{,}000014198 \\ +\ 0{,}000001597 \\ +\ 0{,}000000303 \\ +\ 0{,}000000078 \\ +\ 0{,}000000025 \\ +\ 0{,}000000009 \end{array} \right\}$$

$$= 0{,}000464239$$

Cui si adduntur termini priores seriei modo vulgari sibi invicem additi

$$0{,}003879461$$

habetur series . . 0,004343700

sive, quia septem priores figuræ decimales tantum certæ sunt

$= 0{,}0043437$

Sit 3) series proposita

$$\frac{1}{\sqrt{2}.(\sqrt{2}+\sqrt{3})} + \frac{1}{(\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{2})} + \ldots .$$

de qua §. 2. loquuti sumus, cujus summatio ex priori methodo pæne immensum laborem præbuisset. Quia $\sqrt{2} < \sqrt{3}$, aliquot priorum terminorum modo vulgari sibi invicem addendi sunt. Est

$$\frac{1}{\sqrt{2}\,(\sqrt{2}+\sqrt{3})} = 0{,}2247448$$

$$\frac{1}{(\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{3})} = 0{,}0310102$$

$$\frac{1}{(\sqrt{2}+4\sqrt{3})(\sqrt{2}+5\sqrt{3})} = 0{,}0118983$$

$$\frac{1}{(\sqrt{2}+6\sqrt{3})(\sqrt{2}+7\sqrt{3})} = 0{,}0062561$$

$$\frac{1}{(\sqrt{2}+8\sqrt{3})(\sqrt{2}+9\sqrt{3})} = 0{,}0038515$$

Eorum summa = 0,2777609

Jam superest series

$$\frac{1}{(\sqrt{2}+10\sqrt{3})(\sqrt{2}+11\sqrt{3})} + \frac{1}{(\sqrt{2}+12\sqrt{3})(\sqrt{2}+13\sqrt{3})} + \ldots$$

in qua $b = \sqrt{2} + 10\sqrt{3}$, $d = \sqrt{3}$. Hinc, quia $\frac{b+2}{2d} = \frac{\sqrt{2}+11\sqrt{3}}{2\sqrt{3}}$ > 5, < 6 est, hoc casu expressionis nostræ vix plus quam quinque termini utiliter computantur, quod tamen, ut quinta figura decimalis certa habeatur, sufficiet. Hi sunt, ex expressione non recurrente hic commodius applicanda *)

$$\frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}+10\sqrt{3}} + \frac{1}{4} \cdot \frac{1}{(\sqrt{2}+10\sqrt{3})(\sqrt{2}+12\sqrt{3})}$$

$$+ \frac{\sqrt{3}}{2} \cdot \frac{1}{(\sqrt{2}+10\sqrt{3}) \ldots (\sqrt{2}+14\sqrt{3})}$$

$$+ \frac{3.5.3}{8} \cdot \frac{1}{(\sqrt{2}+10\sqrt{3}) \ldots (\sqrt{2}+16\sqrt{3})}$$

$$+ \frac{3.5.7.3\sqrt{3}}{10} \cdot \frac{1}{(\sqrt{2}+10\sqrt{3}) \ldots (\sqrt{2}+18\sqrt{3})}$$

*) In recurrente enim quovis termino $\sqrt{3}$ habetur, quam computare opus non est. Quomodo ipsi termini, prioribus jam computatis facilius inveniantur, computanti mox in oculos incurret.

$$= \left\{ \begin{array}{r} 0{,}0154086 \\ +\ 0{,}0006011 \\ +\ 0{,}0000811 \\ +\ 0{,}0000181 \\ +\ 0{,}0000054 \end{array} \right\} = 0{,}0161143$$

cui si addideris numerum supra inventum

$$\underline{0{,}2777609}$$

Erit series proposita $= 0{,}2938752$

Est autem $\Delta^{v} < \frac{5.2\sqrt{3}}{\sqrt{2}+11\sqrt{3}} \cdot 0{,}0000054 < 0{,}000005.$ Hinc quinta figura decimalis certa est $S = 0{,}29387$, pro quo tamen, quia, quod sequitur, est $> 0{,}000005$, potius $S = 0{,}29388$ ponendum est.

Ut ejusdem seriei sexta figura decimalis certa sit, addantur tres termini seriei computando modo vulgari, quo facto b in eum modum crescit, ut et 5 termini expressionis, vero longe propiores sint, et 6tus 7mus ve addi possint. Sex priorum seriei terminorum summa supra inventa est . . . 0,2777609

$$\frac{1}{(\sqrt{2}+10\sqrt{3})(\sqrt{2}+11\sqrt{3})} = 0{,}0026080$$

$$\frac{1}{(\sqrt{2}+12\sqrt{3})(\sqrt{2}+13\sqrt{3})} = 0{,}0018824$$

$$\frac{1}{(\sqrt{2}+14\sqrt{3})(\sqrt{2}+15\sqrt{3})} = \underline{0{,}0014224}$$

Itaque 9 prior. term. summa $= 0{,}2836737$

Series, quæ jam superest, incipit a termino

$\frac{1}{(\sqrt{2}+16\sqrt{3})(\sqrt{2}+17\sqrt{3})}$, unde est $b = \sqrt{2}+16\sqrt{3}$ et expressionis nostræ 5 priores termini

$$\frac{1}{2\sqrt{3}} \cdot (2+16\sqrt{3}) + \frac{1}{2} \cdot \frac{1}{(\sqrt{2}+16\sqrt{3})(\sqrt{2}+18\sqrt{3})}$$

$$+ \frac{\sqrt{3}}{2} \cdot \frac{1}{(\sqrt{2}+16)(\sqrt{3}).(\sqrt{2}+20\sqrt{3})}$$

$$+\frac{3\cdot 5\cdot 3}{8}\cdot\frac{1}{(\sqrt{2}+16\sqrt{3})\ldots(\sqrt{2}+22\sqrt{3})}$$

$$+\frac{3\cdot 5\cdot 7\cdot 3\sqrt{3}}{10}\cdot\frac{1}{(\sqrt{2}+16\sqrt{3})\ldots(\sqrt{2}+24\sqrt{3})}$$

$$=\left\{\begin{array}{l} 0{,}0099132 \\ +\ 0{,}0002633 \\ +\ 0{,}0000253 \\ +\ 0{,}0000040 \\ +\ 0{,}0000003 \end{array}\right\} = 0{,}0102061$$

quod supra invenimus $0{,}2836737$

et series $= 0{,}2938798$

Hic, quia $\Delta^{v} < 0{,}0000002$, sexta figura certa est

$s = 0{,}293879$

pro quo tamen, quia sequens figura > 5 est, potius

$S = 0{,}293880$

ponendum est.

§. 12.

Huic etiam summationi observationes quasdam adiicere lubet.

Observatio I. Quibusdam casibus hanc summationem facilius quam priorem procedere vidimus. Hoc autem minime ita intelligendum est, quasi semper aptior methodus sit. Simplicioribus enim in casibus, ubi differentia d parvus numerus est, prior longe facilius procedit. Quod cuivis, qui v. c. seriem

$$\frac{1}{1.3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\ldots\ldots$$

secundo modo summare velit, patebit.

Observatio II. Alterutra harum summationum fere quamvis serierum ad formam generalem propositam pertinentium satis commode summari posse patet, uno tantum genere excepto, si ob b negativum magnus numerus terminorum adest, qui ea seriei forma quæ in utraque summatione adesse debet, non continentur. De hoc

genere infra, postquam summationem seriei numeri finiti terminorum docuerimus, agemus.

Observatio III. Aeque ac prior summatio, ita et hæc altera ad alias etiam series applicari potest. Facile enim patet, series

$$\frac{1}{b.(b+d)(b+2d)}+\frac{1}{(b+2d)(b+3d)(b+4d)}+\ldots\ldots$$

$$\frac{1}{b.(b+d)(b+2d)(b+4d)}+\frac{1}{(b+2d)(b+3d)(b+4d)(b+6d)}+\ldots.$$

et similes, quorum forma generalis hæc est,

$$\frac{1}{b.(b+d)(b+2d)(b+4d)\ldots(b+2kd)}$$

$$+\frac{1}{(b+2d)(b+3d)(b+4d)\ldots.(b+2(k+1)d}+\ldots\ldots,$$

eas cum seriebus §. 9. propositis comparando, similiter summari posse. Quod diligentius excutere itemque inquirere annon aliarum etiam serierum, v. c.

$$\frac{1}{b.(b+d)(b+2d)(b+3d)}+\frac{1}{(b+2d)(b+3d)(b+5d)}+\ldots$$

summatio, simili via, quæ tamen necessario aliquantum difficilior esse debet, inveniri possit, nunc non vacat.

Sectio II. Si series proposita finito terminorum numero constat.

A) Prior methodus.

§. 13.

Seriei propositæ certus terminorum numerus methodorum supra expositarum posteriori directo summari potest. Expressio enim, quam, §. 9. deduximus pro x terminis valet, cui tantum §. 10. eum casum substituimus, quo x numerus infinitus est. Hinc itaque id, quo jam opus habemus, explicare potuerimus. Sed melior est hæcce inspectio. Series certi numeri terminorum

$S =$

$$S=\frac{1}{b.(b+d)}+\frac{1}{(b+2d)(b+3d)}+\ldots+\frac{1}{(b+2(x-1)d)(b+(2x-1)d)}$$

considerari potest ut differentia duarum in infinitum excurrentium

$$\frac{1}{b.(b+d)}+\frac{1}{(b+2d)(b+3d)}+\ldots\ldots=T$$

$$\frac{1}{(b+2xd)(b+(2x+1)d)}+\frac{1}{(b+2(x+1)d)(b+(2x+3)d)}+\ldots=U$$

ut sit $S=T-U$.

Quarum utriusque valor quum assignari queat, differentia quoque innotescet.

Hinc, ut ad singula descendamus, primum hic etiam facere debemus valorem quantitatum b et d esse positivum, quia pro b negativo, d positivo summatio seriei in infinitum excurrentis nondum tradita est, (cf. §. 3.). Quo facto valorem prioris seriei infinitæ T, modo secundum priorem modo secundum posteriorem methodum assignare præstabit, ut ex antecedentibus patet; posterioris autem U valor semper secundum posteriorem methodum inveniri debet. Quum enim prior methodus supponat esse $b-1<2d$, itemque, ut utiliter adhibeatur, esse d (ergo etiam b) numerum parvum, non aliter directo adhiberi potest, nisi $b+2xd$, quod pro b ponitur, numerus parvus, ergo etiam x numerus perparvus sit, i. e. perpauci seriei finitæ termini habeantur. Eo autem casu eos modo vulgari sibi invicem addere omni summationi artificiosæ præstat. Quodsi numerus x magnus est, potest quidem summatio prior adhiberi, sed ita tantum, ut series, antea additis aliquot terminis antecedentibus, in aliam transformetur, in qua $b-1<2d$, qui deinde, peracta summatione, denuo subtractione tollendi essent. Quo facto, quum ii ipsi termini, ex quibus series finita constat, certe major eorum pars, tollendi sint, revera ejus seriei summatio nulla perageretur. Sit v. c. series

$$\frac{1}{1.3}+\frac{1}{5.7}+\ldots\ldots+\frac{1}{97.99}=S$$

quæ considerari potest ut differentia duarum infinitarum

$$\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \ldots\ldots \text{ in inf.} = T$$

$$\frac{1}{101\cdot 103} + \frac{1}{105\cdot 107} + \ldots\ldots \text{ in inf.} = U.$$

Jam, si quis seriem U methodo priori summare velit, quia $b - 1 = 100$ non est $< d = 2$, prius tot termini priores addendi sunt, ut fiat $b - 1 < d$, i. e. $\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \ldots\ldots + \frac{1}{97\cdot 99}$, quibus singulis series finita constat, quæ itaque hoc modo nullatenus summaretur.

Quare semper secunda series infinita U methodo secunda summanda est. Quod si idem primæ seriei accidit, differentia duarum serierum infinitarum ad eandem formulam deducit, quæ §. 9. indicata est.

§. 14.

Ad exempla ut progrediamur, sit

1) series proposita

$$S = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \ldots\ldots + \frac{1}{397\cdot 399}$$

ubi habetur $T = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \ldots\ldots$ in inf.

$$U = \frac{1}{401\cdot 403} + \frac{1}{405\cdot 407} + \ldots\ldots \text{ in inf.}$$

Prior series T facillime methodo priori summatur. Est enim, quod supra vidimus, $= \frac{\pi}{8}$. Posterior U altera methodo summanda est (quia $b = 401$, $d = 2$), ubi expressione recurrente est

$$\mathfrak{A} = \frac{1}{4\cdot 401}$$

$$\mathfrak{B} = \frac{1}{405}\cdot \mathfrak{A}$$

$$\mathfrak{C} = \frac{-4}{409}\cdot \mathfrak{B} \text{ etc.}$$

Hinc octo priores figuræ decimales nullo fere labore ex tribus primis terminis accurate computantur. Est enim

$$\mathfrak{A} = 0{,}000623441396$$
$$\mathfrak{B} = 0{,}000001539361$$
$$\mathfrak{C} = 0{,}000000015055$$
$$U = 0{,}000624995812$$

ubi vitium $\Delta^{III} < \frac{3}{403} \cdot 0{,}000000060$
$< 0{,}0000000005$

ergo octava figura decimalis certa est. Hinc est

$$S = T - U = \left\{ \begin{array}{r} 0{,}3926990817 \\ - 0{,}0006249958 \end{array} \right\}$$
$$= 0{,}39207408.$$

Imo viginti figuræ decimales sine magno labore assignari possunt, et quidem, quod ex consideratione expressionis pro $\Delta^{(f)}$ patet, duodecim terminos computando, ubi etiam nihil impedit, quominus $\frac{1}{401 \cdot 403}$ seriei summandæ primum terminum esse sinas. Est enim $12 < \frac{403}{4}$. Quo certius sit, quod inveniamus, singulorum terminorum 22 figuras decimales computabimus, huncque computum eo usque continuabimus, dum in 22da figura decimali numerus inveniatur. Est autem in expressione recurrente

$$\mathfrak{A} = \frac{1}{4 \cdot 401},\ \mathfrak{B} = \frac{1}{405} \cdot \mathfrak{A},\ \mathfrak{C} = \frac{4}{409} \cdot \mathfrak{B}.$$

$$\mathfrak{D} = \frac{3 \cdot 5 \cdot 2}{4 \cdot 413} \cdot \mathfrak{C},\ \mathfrak{E} = \frac{4 \cdot 7 \cdot 2}{5 \cdot 417} \cdot \mathfrak{D}$$

$$\mathfrak{F} = \frac{5 \cdot 9 \cdot 2}{6 \cdot 421} \cdot \mathfrak{E},\ \mathfrak{G} = \frac{6 \cdot 11 \cdot 2}{7 \cdot 425} \cdot \mathfrak{F}$$

$\mathfrak{H} = \frac{7.13.2}{8.429} . \mathfrak{G}, \ \mathfrak{I} = \frac{8.15.2}{9.433} . \mathfrak{H}.$

$\mathfrak{K} = \frac{9.17.2}{10.437} . \mathfrak{I}, \ \mathfrak{L} = \frac{10.19.2}{11.441} . \mathfrak{K}$

$\mathfrak{M} = \frac{11.21.2}{12.445} . \mathfrak{L}$, et hinc

$$U = \left\{ \begin{array}{r} 0,0006234413965087281795 \\ +\ 0,000015393614728610572 \\ +\ 0,000000150548799301815 \\ +\ 0,00000002733937033326 \\ +\ 0,000000000073420483869 \\ +\ 0,0000000000002616452394 \\ +\ 0,0000000000000116082459 \\ +\ 0,0000000000000006155888 \\ +\ 0,0000000000000000379115 \\ +\ 0,0000000000000000026547 \\ +\ 0,0000000000000000002079 \\ +\ 0,0000000000000000000180 \\ +\ 0,0000000000000000000017 \\ +\ 0,0000000000000000000002 \\ \hline \end{array} \right\}$$

$$= 0,00062499609387206\ 0058$$

$$T = \frac{\pi}{8} = 0,3926990816987241548078$$

$$S = T - U = 0,3920740856048520938 0$$

2) Proposita sit series hæcce

$$\frac{1}{1.2} + \frac{1}{3.4} + \ldots . + \frac{1}{99.100} = S$$

ubi habetur $T = \frac{1}{1.2} + \frac{1}{3.4} + \ldots .$ in inf.

$U = \frac{1}{101.102} + \frac{1}{103.104} + \ldots .$ in inf., $S = T - U$

T priori methodo facillime summatur, estque, ut §. 6:a vidimus $= \log 2 = 0{,}6931471805599\ldots$ U posteriori methodo computanda est. Expressionis inde deducendæ sex terminis opus est, ut 10 priores figuræ decimales inveniantur. Sic habetur

$$U = \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots$$

ubi $\mathfrak{A} = \frac{1}{2.101}$, $\mathfrak{B} = \frac{1}{2.103} \cdot \mathfrak{A}$

$$\mathfrak{C} = \frac{2}{105} \cdot \mathfrak{B}, \ \mathfrak{D} = \frac{3.5}{4} \cdot \frac{1}{107} \mathfrak{C},$$

$$\mathfrak{E} = \frac{4.7}{5} \cdot \frac{1}{109} \cdot \mathfrak{D}, \ \mathfrak{F} = \frac{5.9}{6} \cdot \frac{1}{111}$$

$$U = \left\{ \begin{array}{r} 0{,}004950495049 \\ +\ 0{,}000024031529 \\ +\ 0{,}000000457743 \\ +\ 0{,}000000016042 \\ +\ 0{,}000000000824 \\ +\ 0{,}000000000056 \end{array} \right\}$$

$$= 0{,}004975001243$$

ergo, quia $T = 0{,}693147180560$

est $S = 0{,}6881721793$.

Simili modo 3) series

$$\frac{1}{5.8} + \frac{1}{11.14} + \ldots + \frac{1}{95.98} = S$$

ex differentia supra exhitæ $\frac{1}{5.8} + \frac{1}{11.14} + \ldots$ in inf. $=$

$0{,}0421497577$

et $\frac{1}{101.104} + \frac{1}{107.110} \ldots$ in inf.

quæ posteriori methodo, adhibitis 9 terminis, usque ad decimam figuram decimalem summatur $= 0{,}0016746616$

invenitur $S = 0{,}0104750961$.

4) Proposita series sit

$$\frac{1}{101 \cdot 103} + \frac{1}{105 \cdot 107} + \ldots + \frac{1}{397 \cdot 399} = S$$

ut habeatur

$$\frac{1}{101 \cdot 103} + \frac{1}{105 \cdot 107} + \ldots \text{ in inf.} = T$$

$$\frac{1}{401 \cdot 403} + \frac{1}{405 \cdot 407} + \ldots \text{ in inf.} = U$$

$$S = T - U.$$

Hic utraque series infinita commodius methodo secunda summatur. Decima figura decimalis assignari debeat. Pro serie T est $b = 101$, $d = 2$. Quia ratio quantitatis b ad d permagna est, patet totam seriem commode summari posse, neque opus esse, ut aliquot priorum terminorum separatim methodo vulgari addantur. Est autem in expressione recurrente

$$\mathfrak{A} = \frac{1}{4 \cdot 101}, \quad \mathfrak{B} = \frac{1}{105} \cdot \mathfrak{A}, \quad \mathfrak{C} = \frac{4}{109} \cdot \mathfrak{B},$$

$$\mathfrak{D} = \frac{3 \cdot 5 \cdot 2}{4 \cdot 113} \cdot \mathfrak{C}, \quad \mathfrak{E} = \frac{4 \cdot 7 \cdot 2}{5 \cdot 117} \cdot \mathfrak{D},$$

$$\mathfrak{F} = \frac{5 \cdot 9 \cdot 2}{6 \cdot 121} \mathfrak{E}, \quad \mathfrak{G} = \frac{6 \cdot 11 \cdot 2}{7 \cdot 125} \cdot \mathfrak{F}$$

$$\mathfrak{H} = \frac{7 \cdot 13 \cdot 2}{8 \cdot 129} \cdot \mathfrak{G}, \quad \mathfrak{I} = \frac{8 \cdot 15 \cdot 2}{9 \cdot 133} \cdot \mathfrak{H}$$

$$\mathfrak{K} = \frac{9 \cdot 17 \cdot 2}{10 \cdot 137} \cdot \mathfrak{I}, \text{ et hinc}$$

$$T' = \left\{ \begin{array}{r} 0{,}002475247525 \\ +\ 0{,}000023573786 \\ +\ 0{,}000000865043 \\ +\ 0{,}000000057412 \\ +\ 0{,}000000005496 \\ +\ 0{,}000000000681 \\ +\ 0{,}000000000103 \\ +\ 0{,}000000000018 \\ +\ 0{,}000000000003 \\ +\ 0{,}000000000001 \\ \hline \end{array} \right\} = 0{,}002499750124$$

U hac paragrapho, exemplo primo, inventa est $= 0{,}000624996094$.

Hinc est series proposita $S = 0{,}0018747540$

B) *Altera methodus.*

§. 15.

Superest alia series nostras in infinitum non excurrentes summandi methodus, difficultatibus quidem nonnullis, at, quæ omnino vinci possunt, implicata. Pendet ea a numeris Bernouillianis, quorum ope Eulerus *) serierum harmonicarum numerum finitum terminorum summare docet. Quæ methodus, ad nostras etiam series summandas inservire potest, quum generalis earum forma

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots \frac{1}{(b+2(x-1)d)(b+(2x-1)d)}$$

semper in duas harmonicas dispesci potest:

$$\frac{1}{d} \left\{ \begin{array}{l} \frac{1}{b} + \frac{1}{b+2d} + \frac{1}{b+4d} + \ldots + \frac{1}{b+2(x-1)d} \\ -\frac{1}{b+d} - \frac{1}{b+3d} - \frac{1}{b+5d} - \ldots - \frac{1}{b+(2x-1)d} \end{array} \right\}$$

*) Calcul. differential p. 2. cap. 6. praecip. §. 143. 146.

Habemus autem ex iis, quæ Eulerus demonstravit *).

$$\frac{1}{b} + \frac{1}{b+2d} + \ldots + \frac{1}{b+2(x-1)d} =$$

$$\text{Const.} + \frac{1}{2d}\log(2d(x-1)+b) + \frac{1}{2(2d(x-1)+b)}$$

$$-\frac{\mathfrak{A}}{2}\cdot\frac{2d}{(2d(x-1)+b)^2} + \frac{\mathfrak{B}}{4}\cdot\frac{(2d)^3}{(2d(x-1)+b)^4}$$

$$-\frac{\mathfrak{C}}{6}\cdot\frac{(2d)^5}{(2d(x-1)+b)^6} + \frac{\mathfrak{D}}{8}\cdot\frac{(2d)^7}{(2d(x-1)+b)^8}$$

$$- \ldots \ldots + \ldots \ldots$$

ubi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$.. numeri Bernouilliani sunt, quos per 2, 4, 6.. divisos forma commoda exhibet Eulerus **).

Eodemque modo, substituendo pro b | $b+d$, pro

$\frac{1}{b+d} + \frac{1}{b+3d} + \ldots + \frac{1}{b+(2x-1)d}$ expressio assignari potest.

Posita itaque priore serie $= T$, posteriore $= U$, et ea, quam quærimus, $= S$, habemus

$$S = \frac{T-U}{d}.$$

Constantes hisce seriebus addendæ non ita facili negotio, quam alias plerumque, definiri possunt, quum nullus detur valor τῆς x pro quo series faciliorem formam induant. Quare aliter definiri nequeunt, nisi ut pro certo quodam valore τῆς $x = f$ series et vulgari methodo computetur, et summatoriæ expressionis termini præter constantem assignentur. Utriusque differentia constanti æquatur. Quia autem seriei summatoriæ termini pro magno x longe celerius quam pro parvo decrescant, ut constans assignetur, τῆς x non nimis parvus valor adhiberi debet. Quare Eulerus eum = 10 facere solitus est.

§. 16.

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*) Calculus differentialis l. c. §. 146.

**) l. c. §. 145.

§. 16.

Earum autem serierum, quum ea natura sit, ut priores tantummodo earum termini convergant; posteriores, idque tandem celerius qualibet ratione data, divergant *): magna dubitatio oriri potest, an, et quatenus tuto hæ summationes adhiberi possint? Sed salva res est. Demonstrabimus enim 1) certe pro iis casibus, qui hic usu venire possunt, totam seriei post certum aliquem terminum (ntum) sequentem partem minorem esse eo, termino, qui proxime antecedit (nto); et 2) pro iis valoribus τῆς x, pro quibus operæ pretium erit, vulgari computandi methodo relicta, summationem adhibere, i. e. pro iis, quæ multis terminis constant (pro x numero magno) series hasce eousque convergere, dum termini ita parvi habeantur, quos negligere licebit **).

*) vd. Eulerum l. c. §. 129. Ex eo, quod ibi demonstratum est, numeros $\alpha, \beta\ \gamma$ .. tandem celerius quavis progressione geometrica crescere; idem de numeris Bernouillianis ipsis indeque etiam de terminis nostræ seriei, valere, facile deducitur.

**) Prius num et quomodo demonstrari possit, quum hancce commentationem Hafniam misi, mihi nondum innotuit. Quare tunc dubitanter de hac summationis methodo loquutus sum. Nunc autem beneficio v. cel. J. F. *Pfaffii et* sagacis Erchingeri, (qui, ante paucos annos rusticus, nunc mathesin apprime sciens, eam in Lyceo Tubingensi docet), hoc quoque dubium evanuit. Ille enim, in literis humanissime ad me datis, monuit, celebrem Legendre in opere inscripto exercices de calcul integral p. 2. p. 267 aliquot similes series, quæ item a numeris Bernouillianis pendent, primum convergentes, mox divergentes, quas vocat *series semi-convergentes*, considerasse, deque iis asseruisse, quod eousque, donec divergentes fiant, adhiberi possint. Quae asserta, cum Eulerianis convenientia, quum probatione egeant, cel. Pfaffius thema probandum ita definivit: in seriebus semiconvergentibus, quarum termini alternatim positivi negativique sint, terminorum divergentium summam minorem esse ultimo termino convergente. Quod sic evenire de serie $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ in inf.,

I.) Primum respectu omnium serierum, quas Eulerus in cap. 5to lib. 2di calculi differentialis summare docet, quibusque nostra quoque series subest, sic demonstratur. Generalis earum summationum formula, signis ab Eulero ipso adhibitis, est:

$$Sz = \int z dx + \tfrac{1}{2} z + A. \frac{dz}{dx} - B. \frac{d^3 z}{dx^3} + C. \frac{d^5 z}{dx^5} - \ldots \text{*)}$$

quæ formula, quovis casu, ut et constans in integrali contenta assignetur, bis applicari debet. Faciamus enim summationem pro $x = g$ effici debere; eamque, modo computandi vulgari, pro $x = f$ inventam esse, et significare

$$[f]\left(\int z dx + \tfrac{1}{2} z + A \frac{dz}{dx} - B \frac{d^3 z}{dx^3} \mp C \frac{d^5 z}{dx^5} - \ldots\right)$$

expressionem summatoriam pro $x = f$, absque tamen constante; erit

$$\text{const.} = Sf - [f]\left(\int z dx + \tfrac{1}{2} z + A. \frac{dz}{dx} - B. \frac{d^3 z}{dx^3} + C. \frac{d^5 z}{dx^5} - \ldots\right)$$

et, si præfixum $[g]$ simili significatione adhibetur, erit, quæ quæritur, summa,

$$Sg = Sf + [g]\left(\int z dx + \tfrac{1}{2} z + A. \frac{dz}{dx} - B. \frac{d^3 z}{dx^3} + C. \frac{d^5 z}{dx^5} - \ldots\right)$$
$$- [f]\left(\int z dx + \tfrac{1}{2} z + A. \frac{dz}{dx} - B. \frac{d^3 z}{dx^3} + C. \frac{d^5 z}{d^5 x} - \ldots\right)$$

Quam seriem summatoriam utramque faciamus ad ntum usque terminum $\mp N \frac{d^{2n-1} z}{dx^{2n-1}}$, computatam esse, ita ut in utraque neglecti sint

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quae pro $x > 1$ e semiconvergentibus est, facili negotio demonstravit, et generatim demonstrari posse suspicatus, neque tamen, aliis negotiis distractus, quo tempore literas mihi dedit, perfecerat. Quod de omnibus iis seriebus, quae Eulerianae summationi in Calc. Diff. tom. 2. cap. 5. traditae subsunt, quarum etiam ea est, de qua hic agimus, Erchingerus methodo ingeniosa in contextu exhibita Pfaffianae haud absimili, perfecit.

*) Eulerus l. c. §. 110.

$$\pm \left(N'.\frac{d^{2n+1}z}{dx^{2n+1}} - N''.\frac{d^{2n+3}z}{dx^{2n+3}} + N'''.\frac{d^{2n+5}z}{dx^{2n+5}} - \ldots\ldots\right)$$

Qui, quum ex iis, quæ Eulerus alibi tradidit *), sint

$$A = \frac{a}{2\pi^2} = \frac{1}{2\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$= 2\left(\frac{1}{(2\pi)^2} + \frac{1}{(4\pi)^2} + \frac{1}{(6\pi)^2} + \frac{1}{(8\pi)^2} + \ldots\ldots\right)$$

$$B = \frac{b}{2^3\pi^4} = \frac{1}{2^3\pi^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots\right)$$

$$= 2\left(\frac{1}{(2\pi)^4} + \frac{1}{(4\pi)^4} + \frac{1}{(6\pi)^4} + \frac{1}{(8\pi)^4}\ldots\ldots\right)$$

. . . .

$$N = \frac{n}{2^{2n-1}.\pi^{2n}} = \frac{1}{2^{2n-1}.\pi^{2n}}.\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \ldots\right)$$

$$= 2\left(\frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \frac{1}{(6\pi)^{2n}} + \frac{1}{(8\pi)^{2n}} + \ldots\right)$$

. . . .

erunt $= \pm\, 2\,N$, si est

$$N = \left\{\begin{array}{l}
\frac{1}{(2\pi)^{2n+2}}.\frac{d^{2n+1}z}{dx^{2n+1}} + \frac{1}{(4\pi)^{2n+2}}.\frac{d^{2n+1}z}{dx^{2n+1}} + \ldots + \frac{1}{(2m\pi)^{2n+2}}.\frac{d^{2n+1}z}{dx^{2n+1}} + \ldots \\
- \frac{1}{(2\pi)^{2n+4}}.\frac{d^{2n+3}z}{dx^{2n+3}} - \frac{1}{(4\pi)^{2n+4}}.\frac{d^{2n+3}z}{dx^{2n+3}} - \ldots - \frac{1}{(2m\pi)^{2n+4}}.\frac{d^{2n+3}z}{dx^{2n+3}} - \ldots \\
+ \frac{1}{(2\pi)^{2n+6}}.\frac{d^{2n+5}z}{dx^{2n+5}} + \frac{1}{(4\pi)^{2n+6}}.\frac{d^{2n+5}z}{dx^{2n+5}} + \ldots + \frac{1}{(2m\pi)^{2n+6}}.\frac{d^{2n+5}z}{dx^{2n+5}} + \\
- \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}$$

sive, si hujus expressionis eas partes, quæ verticales series efficiunt, $y', y'' \ldots y^{(m)} \ldots$ vocaveris,

$$N = y' + y'' + \ldots + y^{(m)} + \ldots\ldots$$

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*) l. c. §. 125.

Quae quantitates singulæ ea qualitate gaudent, ut, si eas bis differentiaveris, easdem series, demto tantum primo termino, mutatis signis, adjectoque factore $(2\pi)^2$, $(4\pi)^2 \ldots (2m\pi)^2 \ldots$ obtineas; sive, ut sit

$$\frac{d^2y'}{dx^2} = -(2\pi)^2 . y' + \frac{1}{(2\pi)^{2n+1}} \cdot \frac{d^{2n+1}z}{dx^{2n+1}}$$

$$\frac{d^2y''}{dx^2} = -(4\pi)^2 . y'' + \frac{1}{(4\pi)^{2n+1}} \cdot \frac{d^{2n+1}z}{dx^{2n+1}}$$

. . . .

$$\frac{d^2y^{(m)}}{dx^2} = -(2m\pi)^2 . y^{(m)} + \frac{1}{(2m\pi)^{2n+1}} \cdot \frac{d^{2n+1}z}{dx^{2n+1}}$$

. . . .

Quarum expressionum si generalem pro $y^{(m)}$ consideraveris, ubique, ut calculi facilius subducantur, pro $\frac{1}{(2m\pi)^{2n}} \cdot \frac{d^{2n+1}z}{dx^{2n+1}}$ posueris X, habebis hanc æquationem differentialem secundi gradus:

$$\frac{d^2y^{(m)}}{dx^2} + (2m\pi)^2 . y^{(m)} - X = 0$$

hincque *)

$$y^{(m)} = \left\{ \begin{array}{l} \left(C - \frac{\int X \cos 2m\pi x . dx}{2m\pi}\right) \sin 2m\pi x \\ + \left(C + \frac{\int X \sin 2m\pi x . dx}{2m\pi}\right) \cos 2m\pi x \end{array} \right\}$$

Quia autem nostro casu m. x, quippe indicem terminorum significantes, semper numeri integri sunt; indeque semper $\sin 2m\pi x = 0$, $\cos 2m\pi x = 1$: pro hisce integralibus nostro casu poni potest

$$y^{(m)} = \frac{\int X \sin 2m\pi x . dx}{2m\pi} + C'.$$

Quum autem, a signis abstrahendo, semper $\sin 2m\pi x \leqq 1$, erit quoque, item a signis abstrahendo,

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*) vd. v. cel. Bohnenbergeri höhere Analysis §. 174. num. 3.

$$\frac{\int X \sin 2m\pi . x . dx}{2m\pi} \leqq \frac{\int X dx}{2m\pi}$$

i. e. si loco X ejus valorem restitueris et vere integraveris

$$\leqq \frac{1}{(2m\pi)^{2n}} \cdot \frac{d^{2n}z}{dx^{2n}}$$

et hinc, iterum a signis abstrahendo, (i. e. absolutas quantitates utriusque termini sibi invicem addendo)

$$y^{(m)} \leqq \frac{1}{(2m\pi)^{2n}} \cdot \frac{d^{2n}z}{dx^{2n}} + C';$$

et, substituendo loco m successive $1, 2..m$, iterumque a signis abstrahendo,

$$2N = 2(y' + y'' + \ldots + y^{(m)})$$

$$\leqq 2\left(\frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \ldots + \frac{1}{(2m\pi)^{2n}}\right) \cdot \frac{d^{2n}z}{dx^{2n}} + C''$$

$$\leqq N . \frac{d^{2n}z}{dx^{2n}} + C''$$

Hinc, si ad eas summationes revertimur, quæ, ut S obtineatur, adhiberi debent, constantes evanescunt, et habetur, si utraque usque ad ntum terminum computata est, vitium, iterum a signis abstrahendo

$$\leqq N\left([g]\frac{d^{2n}z}{dx^{2n}} - [f]\frac{d^{2n}z}{dx^{2n}}\right) + C'' - C''$$

sive, quia $f < g$ et pro minori x vitium majus esse soleat, ut expressio plerumque positiva evadat

$$\leqq N\left([f]\frac{d^{2n}z}{dx^{2n}} - [g]\frac{d^{2n}z}{dx^{2n}}\right).$$

Quam expressionem, pro quovis valore τοῦ z facile cum ultimo termino seriei, quem adhuc computasti, $N\frac{d^{2n-1}z}{dx^{2n-1}}$, comparabis. Nostro casu, ubi est

$$z = \frac{1}{2d(x-1)+b+\left\{\begin{matrix} o \\ d \end{matrix}\right\}}$$ habetur *)

$$N\frac{d^{2n}z}{dx^{2n}} = -N\cdot\frac{d^{2n-1}z}{dx^{2n-1}}\cdot\frac{2n}{2d(x-1)+b+\left\{\begin{matrix} o \\ d \end{matrix}\right\}}$$

ergo, dum $n < x-1$ — quod semper eveniet, quia nemo tam stolidus erit, ut seriei summatoriæ, cujus singuli termini computatu difficiliores esse solent terminis seriei originariæ, fere eundem numerum computet, quam ad quem usque originaria illa extenditur — abstrahendo a signis

$$N\cdot\frac{d^{2n}z}{dx^{2n}} < N\frac{d^{2n-1}z}{dx^{2n-1}}.$$

Sic patet, in nostris certe seriebus, iisque casibus, qui soli usu venient, vitium semper minus esse ultimo, cujus in computando adhuc rationem habuisti, termino.

II. Quæ usquedum demonstravimus, ut utiliter summatio nostra adhibeatur, omnino sufficiunt, siquidem in computanda ea semper ad terminum ita parvum pervenimus, ut is, sive omnes sequentes, qui eo non majores sunt, tuto negligi possit.

Quod sæpissime evenire, ea exempla docent, quæ Eulerus computavit; semper rem ita disponi posse, ut eveniat, nunc demonstrabimus.

Est autem, ex iis quæ Eulerus **) docuit, si facis ν ntam, $'\nu(n-1)$ tam ***) ex serie quantitatum $\alpha, \beta, \gamma \ldots$,

$$\frac{4^{\nu}}{2n(2n+2)}\cdot\frac{\pi^2}{'\nu} = \frac{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots}{1+\frac{1}{2^{2n-2}}+\frac{1}{3^{2n-2}}+\ldots}$$

*) Superius (*o*) valet pro prima, inferius (*d*) pro posteriori serie summatoria.

**) l. l. §. 129.

***) Eulerus nominat φ, ψ ntam et $n+1$ tam quantitatem.

i. e. quia

$$\frac{1}{2^{2n}}, \frac{1}{3^{2n}} \ldots < \frac{1}{2^{2n-2}}, \frac{1}{3^{2n-2}} \ldots\ldots$$

$$\frac{4\nu}{2n(2n+1)} \cdot \frac{\pi^2}{'\nu} < 1,$$

sive $\frac{\nu}{'\nu} < \frac{2n.(2n+1)}{4\pi^2}$.

et, quum numeri Bernouilliani

$$\mathfrak{N} = \frac{\nu}{2n+1}, \ '\mathfrak{N} = \frac{'\nu}{2n-1}$$

et in seriebus nostris summatoriis, termini ab hisce numeris Bernouilianis pendentes, si a signis abstraxeris,

$$N = \frac{\mathfrak{N}}{2n} \cdot \frac{(2d)^{2n-1}}{\left(2d(x-1)+b+\left\{\begin{matrix}o\\d\end{matrix}\right\}\right)^{2n}}$$

$$'N = \frac{'\mathfrak{N}}{2n-2} \cdot \frac{(2d)^{2n-3}}{\left(2d(x-1)+b+\left\{\begin{matrix}o\\d\end{matrix}\right\}\right)^{2n-2}}$$

$$\frac{N}{'N} < \frac{d^2}{\pi^2} \cdot \frac{(2n-1)(2n-2)}{\left(2d(x-1)+b+\left\{\begin{matrix}o\\d\end{matrix}\right\}\right)^2}$$

Hic si facis $x \geqq \frac{\sqrt{(n-1)(2n-1)}}{\pi} - \frac{b+\left\{\begin{matrix}o\\d\end{matrix}\right\}}{2d} + 1$

erit $\frac{N}{'N} \leqq \frac{1}{2}$.

Si autem est $\frac{N}{'N} \leqq \frac{1}{2}$, majori ratione,

(quum $\frac{'N}{''N} < \frac{d^2}{\pi^2} \cdot \frac{(2n-3)(2n-4)}{\left(2d(x-1)+b+\left\{\begin{matrix}o\\d\end{matrix}\right\}\right)^2} < \frac{d^2}{\pi^2} \cdot \frac{(2n-1)(2n-2)}{\left(2d(x-1)+b+\left\{\begin{matrix}o\\d\end{matrix}\right\}\right)^2}$,

et sic porro,)

erit $\frac{'N}{''N} < \frac{1}{2}, \frac{''N}{'''N} < \frac{1}{2} \ldots\ldots \frac{B}{A} < \frac{1}{2}$, (ubi A, B primum et secun-

dum a Bernouillianis numeris pendentem terminum serierum nostrarum summatoriarum significant.

Hinc habetur ultimus, qui in computum ingressus est, terminus, $N < (\frac{1}{2})^{n-1}. A.$

Jam si nobis propositum est efficere, ut hic ultimus terminus, similiterque vitium, quod admittimus, minus sit certa quadam quantitate, $< \nu$, fiat $(\frac{1}{2})^{n-1}. A \gtreqless \nu$ i. e. $(n-1)\log\frac{1}{2} + \log A \gtreqless \log \nu$

$$n \gtreqless \frac{\log \frac{\nu}{A}}{\log \frac{1}{2}} + 1$$

Quod si ergo numerum terminorum seriei summatoriæ quos computes (n), ex hacce æquatione definias; et numerus terminorum seriei originariæ (x) superiori æquationi satisfacit; semper ultimus terminus, ideoque et vitium commissum minus erit quantitate data (ν). Quod itaque, si hic numerus sat magnus est, quales numquam non usu venire solent, semper fieri potest. Quæ ad dubia contra methodum hancce oriunda generatim removenda sufficient; quovis autem casu speciali, prioribus certe terminis, convergentia ea, quam hic generatim consideravimus, longe major locum habere solet, ita ut, quisnam valor τῆς x in constante assignanda assumi debeat, et quot termini seriei computandæ sufficiant, ne vitium plus quam tolerabile sit, potius experimentis facilibus, quam ex principiis modo deductis inveniatur.

§. 17.

Superest, ut hujus quoque summationis exempla addiiciamus.

Exemplum I. Sit series proposita.

$$\frac{1}{1.3} + \frac{1}{5.7} + \ldots + \frac{1}{397.399}$$

quæ usque ad decimam figuram decimalem exacte computari debeat.

series

Series generalior

$$\frac{1}{1.3}+\frac{1}{5.7}+\ldots+\frac{1}{(4x-3)(4x-1)} \text{ est}$$

$$=\left\{\begin{matrix}\frac{1}{1}+\frac{1}{5}+\ldots+\frac{1}{4x-3}\\ -\frac{1}{3}-\frac{1}{7}-\ldots-\frac{1}{4x-1}\end{matrix}\right\}$$

Quarum prior est

$$=\text{Const.}+\tfrac{1}{4}\log(4x-3)+\frac{1}{2(4x-3)}-\frac{\mathfrak{A}}{2}\cdot\frac{4}{(4x-3)^2}$$

$$+\frac{\mathfrak{B}}{4}\cdot\frac{4^3}{(4x-3)^4}-\frac{\mathfrak{C}}{6}\cdot\frac{4^5}{(4x-3)^6}+\frac{\mathfrak{D}}{8}\cdot\frac{4^7}{(4x-3)^8}-\ldots$$

Constans ut definiatur, sit $x=10$.

Quo facto habetur

$$\text{Const.}=\tfrac{1}{1}+\tfrac{1}{5}+\tfrac{1}{9}+\ldots+\tfrac{1}{37}$$

$$-\left(\tfrac{1}{4}\log 37+\frac{1}{2.37}-\frac{\mathfrak{A}}{2}\cdot\frac{4}{37^2}+\ldots\right)$$

Est autem

$\frac{1}{1} = 1$

$\frac{1}{5} = 0{,}2$

$\frac{1}{9} = 0{,}111111111111$

$\frac{1}{13} = 0{,}076923076923$

$\frac{1}{17} = 0{,}058823529412$

$\frac{1}{21} = 0{,}047619047619$

$\frac{1}{25} = 0{,}04$

$\frac{1}{29} = 0{,}034482758621$

$\frac{1}{33} = 0{,}030303030303$

$\frac{1}{37} = 0{,}027027027027$

$1{,}626289581016$

$$\tfrac{1}{2}\log 37 = 0{,}902729478161$$
$$\frac{1}{2\cdot 37} = 0{,}013513513513$$
$$\frac{\mathfrak{B}}{4}\cdot\frac{4^3}{37^4} = 0{,}000000284572$$
$$\frac{\mathfrak{D}}{8}\cdot\frac{4^7}{37^8} = 0{,}000000000019$$
$$-\frac{\mathfrak{A}}{2}\cdot\frac{4}{37^2} = -0{,}000243486730$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{4^5}{37^6} = -0{,}000000001584$$
$$0{,}915999787951$$

Qui numerus, si a summa fractionum $\frac{1}{1} + \ldots + \frac{1}{37}$ supra inventarum subtrahitur, remanet

$$\text{Const.} = 0{,}710289793065$$

Jam, faciendo $x = 100$, quo casu series nostra $\frac{1}{1} + \frac{1}{5} + \ldots + \frac{1}{397}$ prodit, habetur

$$\text{Const.} = 0{,}710289793065$$
$$\tfrac{1}{2}\log 397 = 1{,}495984070172$$
$$\frac{1}{2\cdot 397} = 0{,}001259445844$$
$$\frac{\mathfrak{B}}{4}\cdot\frac{4^3}{397^4} = 0{,}000000000021$$
$$2{,}207533309102$$
$$-\frac{\mathfrak{A}}{2}\cdot\frac{4}{397^2} = -0{,}000002114936$$
$$\text{Ergo series} = 2{,}207531194166$$

Quod ita exactum esse, ne vitium decimam figuram decimalem transcendat, secundum ea quae §° proxime antecedente deduximus, inde patet, quod ultimus, quem adhibuimus, terminus in decima figura deficit.

Posterior series eodem modo est

$$= \text{Const.} + \tfrac{1}{4}\log(4x-1) + \frac{1}{2.(4x-1)} - \frac{\mathfrak{A}}{2}\cdot\frac{4}{(4x-1)^2}$$

$$+ \frac{\mathfrak{B}}{4}\cdot\frac{4^3}{(4x-1)^4} - \frac{\mathfrak{C}}{6}\cdot\frac{4^5}{(4x-1)^6} + \frac{\mathfrak{D}}{8}\cdot\frac{4^7}{(4x-1)^8} - \ldots$$

Constans ut definiatur, sit $x = 10$. Sic habetur

$$\text{Const.} = \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{11} + \ldots + \tfrac{1}{39}$$

$$- \left(\tfrac{1}{4}\log 39 + \frac{1}{2.39} - \frac{\mathfrak{A}}{2}\cdot\frac{4}{39^2} + \ldots\right)$$

$\frac{1}{3} = 0{,}333333333333$
$\frac{1}{7} = 0{,}142857142857$
$\frac{1}{11} = 0{,}090909090909$
$\frac{1}{15} = 0{,}066666666667$
$\frac{1}{19} = 0{,}052631578947$
$\frac{1}{23} = 0{,}043478260869$
$\frac{1}{27} = 0{,}037037037037$
$\frac{1}{31} = 0{,}032258064516$
$\frac{1}{35} = 0{,}028571428571$
$\frac{1}{39} = 0{,}025641025641$

$0{,}853383629347$

$\frac{1}{4}\log 39 = 0{,}915890411532$

$\frac{1}{2.39} = 0{,}012820512820$

$\frac{\mathfrak{B}}{4}\cdot\frac{4^3}{39^4} = 0{,}000000230537$

$\frac{\mathfrak{D}}{8}\cdot\frac{4^7}{39^8} = 0{,}000000000013$

$0{,}928711154902$

$-\frac{\mathfrak{A}}{2}\cdot\frac{4}{39^2} = -0{,}000219154065$

$$-\frac{\mathfrak{C}}{6}\cdot\frac{4^5}{39^6}=-0,000000001155$$

0,928491999682.

Hunc numerum a summa singularum fractionum supra inventa subtrahendo, fit

Const. = - 0,075108370335

si jam fit $x = 100$, ut series

$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\ldots\ldots+\frac{1}{399}$ prodeat, habemus

$$\tfrac{1}{4}\log 399 = 1,497240354222$$

$$\frac{1}{2\cdot 399} \quad = 0,001253132832$$

$$\frac{\mathfrak{B}}{4}\cdot\frac{4^3}{399^4} \quad = 0,000000000021$$

1,498493487075

Constans = - 0,075108370333

$$-\frac{\mathfrak{A}}{2}\cdot\frac{4}{399^2} = -0,000002093789$$

secunda series = 1,423383022953 quæ ob easdem rationes, quam quæ ad priorem pertinent, satis exacta est.

Quæ si subtrahitur a prima

2,207531194166

restat 0,784148171213;

cujus dimidia pars

0,392074085606

omnino convenienter cum eo, quod supra, alia methodo adhibita, invenimus.

Exemplum II. Series proposita sit

$$\frac{1}{1.2}+\frac{1}{3.4}\ldots+\frac{1}{99.100}=\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{99}-\frac{1}{2}-\frac{1}{4}-\ldots\ldots-\frac{1}{100},$$ quam usque ad octavam figuram decimalem computare velis.

Prioris seriei forma generalior

$\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{2x-1}$ est

$$= \text{Const.} + \tfrac{1}{2}\log(2x-1) + \frac{1}{2\cdot(2x-1)} - \frac{\mathfrak{A}}{2}\cdot\frac{2}{(2x-1)^2}$$

$$+ \frac{\mathfrak{B}}{4}\cdot\frac{2^3}{(2x-1)^4} - \frac{\mathfrak{C}}{6}\cdot\frac{2^5}{(2x-1)^6} + \ldots\ldots$$

Faciendo et hic, ut constans inveniatur, $x = 10$, habetur

$$\text{Const.} = \tfrac{1}{1} + \tfrac{1}{3} + \ldots + \tfrac{1}{19}$$

$$-\left(\tfrac{1}{2}\log 19 + \frac{1}{2.19} - \frac{\mathfrak{A}}{2}\cdot\frac{2}{19^2} + \ldots\right)$$

$\frac{1}{1} = 1$

$\frac{1}{3} = 0{,}3333333333$

$\frac{1}{5} = 0{,}2$

$\frac{1}{7} = 0{,}1428571429$

$\frac{1}{9} = 0{,}1111111111$

$\frac{1}{11} = 0{,}0909090909$

$\frac{1}{13} = 0{,}0769230769$

$\frac{1}{15} = 0{,}0666666667$

$\frac{1}{17} = 0{,}0588235294$

$\frac{1}{19} = 0{,}0526315789$

$2{,}1332555302$

$\frac{1}{2}\log 19 = 1{,}4722194896$

$\frac{1}{2.19} = 0{,}0263157894$

$\frac{\mathfrak{B}}{4}\cdot\frac{2^3}{19^4} = 0{,}0000005115$

$1{,}4985357905$

$-\frac{\mathfrak{A}}{2}\cdot\frac{2}{19^2} = -0{,}0004616805$

$-\frac{\mathfrak{C}}{6}\cdot\frac{2^5}{19^6} = -0{,}0000000027$

$1{,}4980741073$

$$-\frac{\mathfrak{C}}{6}\cdot\frac{1}{2000000}=-0{,}0000000020$$

$$1{,}5224498848.$$

Hunc numerum a summa fractionum 2,133... subtrahendo, remanent
Const. = 0,6351814228.

Jam, ut series $\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{99}$ prodeat, faciendo $x=50$, habetur

$$\begin{aligned}
\text{Const.} &= 0{,}6351814228\\
\tfrac{1}{2}\log 99 &= 2{,}2975599251\\
\frac{1}{2\cdot 99} &= 0{,}0050505050\\
\frac{\mathfrak{B}}{4}\cdot\frac{2^3}{99^4} &= 0{,}0000000007\\
&\overline{2{,}9377918536}\\
-\frac{\mathfrak{A}}{2}\cdot\frac{2}{99^2} &= 0{,}0000170051\\
\text{series ipsa} &= 2{,}9377748485
\end{aligned}$$

Posterioris seriei forma generalis hæc est

$$\tfrac{1}{2}+\tfrac{1}{4}+\ldots+\frac{1}{2x}\ (\text{pro } b=2,\ d=2)$$

$$=\text{Const.}+\tfrac{1}{2}\log 2x+\frac{1}{2.2x}-\frac{\mathfrak{A}}{2}\cdot\frac{2}{(2x)^2}+\frac{\mathfrak{B}}{4}\cdot\frac{2^3}{(2x)^4}-\frac{\mathfrak{C}}{6}\cdot\frac{2^5}{(2x)^6}+\ldots$$

$$=\text{Const.}+\tfrac{1}{2}\log 2x+\frac{1}{4x}-\frac{\mathfrak{A}}{2}\cdot\frac{1}{2x^2}+\frac{\mathfrak{B}}{4}\cdot\frac{1}{2.x^4}-\frac{\mathfrak{C}}{6}\cdot\frac{1}{2x^6}+\ldots$$

Fiat et hic primum, ut constans inveniatur, $x=10$. Habetur

$$\begin{aligned}
\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\ldots+\tfrac{1}{20} &= 1{,}4644841279\\
\tfrac{1}{2}\log 20 &= 1{,}4978661368\\
\tfrac{1}{40} &= 0{,}025\\
\frac{\mathfrak{B}}{4}\cdot\frac{1}{20000} &= 0{,}0000004167\\
&\overline{1{,}5228665535}\\
-\frac{\mathfrak{A}}{2}\cdot\frac{1}{200} &= -0{,}0004166667\\
-\frac{\mathfrak{C}}{6}\cdot\frac{1}{2000000} &= -0{,}000000020\\
&\overline{1{,}5224498848}
\end{aligned}$$

Qui numerus si a prius invento tollitur, remanet:

Const. $= - 0{,}0579657578$

Fiat nunc, pro invenienda serie ipsa,

$x = 50$, est $\frac{1}{2}\log. 10 = 2{,}3025850930$

$$\frac{1}{200} = 0{,}005$$

$$\text{Constans} = - 0{,}0579657578$$

$$- \frac{\mathfrak{A}}{2} \cdot \frac{2}{(100)^2} = - 0{,}0000166667$$

$$\text{series altera} = 2{,}2496026692$$

$$\text{Quæ, si a priori} = 2{,}9377748485$$

subtrahitur, remanet $0{,}6881721793$.

Quibus in computationibus vitium, quod admittitur, quia ultimi termini in septima figura decimali deficiunt, sat parvum est. Convenit quoque hic numerus exacte cum eo, quem supra invenimus.

§. 18.

Coronidis loco hic quoque observationes aliquot adiicere lubet.

I.) Computus hic adhibendus, ut facilius procedat, statim in constante invenienda sat magnus valor pro quantitate x adsumi debet. Qui enim dum augetur, numerus terminorum seriei summatoriæ sibi invicem addendorum, valde minuitur; quod compendium, quia hi termini computatu difficiliores sunt, eo minime compensatur, quod aliquot termini seriei originariæ amplius computandi sunt. Quod qui clarius intelligere velit, faciat periculum constantes §° præcedente exhibitas e valore ~~m~~ $x = 5$ invenire.

II.) Licet series summatoriæ, quæ hic adhibentur celerrime convergant; tamen hac methodo vix umquam celerius et facilius, quam methodo antea exhibita, progredi licebit. Inde enim quod ad quamque seriem summandam quatuor summatoriæ series adhibendæ sunt;

singulique earum termini computatu difficiliores sunt, tanta molestia computanti oritur, ut jam convergentiæ celeritas parum efficiat. Neque illa molestia per utiles terminorum quorundam additiones aliasve summationes minui posse videtur.

Sectio III. De summatione unicae formae seriei propositae, vel in infinitum excurrentis vel non excurrentis, in superioribus non contentae.

§. 19.

Superest unicus casus in superioribus nondum resolutus, si nempe in serie

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots$$

vel finita vel infinita b numerus negativus isque magnus, d positivus est, vel, quod ad idem redit, b positivus, d negativus. Hoc vidimus, quomodo pars hujus seriei præsertim infinitæ summari possit, quum, posito $b+2md=\beta$ ubi m numerus ita magnus est, ut $b+2md$ positivum $b+2(m-1)d$ adhuc negativum sit, ii termini, qui a $\frac{1}{\beta.(\beta+d)}$ incipiunt, summari possunt. Terminorum quidem, qui hosce antecedunt, si parvus numerus est, eorum summa haud difficulter modo vulgari assignatur; si autem magnus numerus, ut additiones nimis multæ et operosæ evitentur, summatio desiderari potest. Quæ haud ita difficulter invenietur. Termini enim illi

$$\frac{1}{b.(b+d)} + \frac{1}{(b+2d)(b+3d)} + \ldots\ldots$$
$$+ \frac{1}{(b+2(m-2)d)(b+(2m-3)d)} + \frac{1}{(b+2(m-1)d)(b+(2m-1)d)}$$

quia $b+2md=\beta$, sic etiam exprimi possunt

$$\frac{1}{(\beta-2md)(\beta-(2m-1)d)} + \frac{1}{(\beta-2(m-1)d)(\beta-(2m-3)d)}$$
$$+\ldots + \frac{1}{(\beta-4d)(\beta-3d)} + \frac{1}{(\beta-2d)(\beta-d)}$$

Jam

Jam, quia $b+2md=\beta=$ numeros positivos >0; $b+2(m-1)d$, $\beta-2d=$ num. negativ. <0: $\beta-d$, ideoque etiam terminus $\frac{1}{(\beta-2d)(\beta-d)}$ utrum negativus an positivus sit, in ambiguo est. Quo paulisper seposito, reliqui termini singuli positivi sunt, quippe constantes bonis factoribus negativis. Quum enim $\beta-2d$ negativum sit, hoc magis adhuc $\beta-3d$, $\beta-4d$.... $\beta-2md$ esse debet. Ut itaque seriei forma etiam positiva fiat, sic scribatur

$$\frac{1}{(2md-\beta)((2m-1)d-\beta)}+\frac{1}{((2m-2)d-\beta)((2m-3)d-\beta)}+\ldots+\frac{1}{(4d-\beta)(3d-\beta)}$$

Facto tandem $3d-\beta=\gamma$, et ordine seriei converso, fit

$$\frac{1}{\gamma\cdot(\gamma+d)}+\frac{1}{(\gamma+2d)(\gamma+3d)}+\ldots+\frac{1}{(\gamma+2(m-2)d)(\gamma+(2m-3)d)}$$

quæ ex iis, quæ sectione secunda exposui, semper summari potest. Addito deinde termino aliquantisper seposito $\frac{1}{(\beta-2d)(\beta-d)}$ summata serie $\frac{1}{\beta.(\beta+d)}+\frac{1}{(\beta+2d)(\beta+3d)}+\ldots$ tota quæstio resoluta est

Casu, quo et $\beta-d$ numerus negativus, ideoque $\frac{1}{(\beta-d)(\beta-2d)}$ positivum est, hic terminus statim cum reliquis jungi potest, quo facto pro $\gamma \mid \beta$, pro $\gamma+2(m-2)d \mid \beta+2(m-1)d$ venit, ut sit summandum

$$\frac{1}{\beta.(\beta+d)}+\frac{1}{(\beta+2d)(\beta+3d)}+\ldots+\frac{1}{(\beta+2(m-1)d)(\beta+(2m-1)d)}$$
$$+\frac{1}{\beta.(\beta+d)}+\frac{1}{(\beta+2d)(\beta+3d)}+\ldots\ldots$$

Exempla hæc habe

1) Sit $b=-399$, $d=2$, series in infinitum excurrens

$$S=\frac{1}{-399\cdot-397}+\frac{1}{-395\cdot-393}+\ldots\ldots \text{ in inf.}$$

Facile patet esse $m=100$, quo facto $b+2md=\beta=400-399=1$, $\beta-1$ adhuc negativum est, ut series in hasce duas dispescatur

$$\frac{1}{1.3}+\frac{1}{5\cdot7}+\ldots\ldots+\frac{1}{397\cdot399}+\frac{1}{1\cdot3}+\frac{1}{5\cdot7}+\ldots\ldots \text{ in inf.}$$

quam utramque jam supra summavimus (vide §. 14.), ut sit

$S=0{,}78477316730357624860$

2) $b=-98$, $d=3$, series in infinitum excurrens

$$S=\frac{1}{-98.-95}+\frac{1}{-92.-89}+\ldots\ldots \text{ in inf.}$$

Hic esse debet $m=17$; quia tunc $b+2md=\beta=102-98=4$,

$b+2(m-1)d=96-98$ adhuc negativum est. Quo facto, quia $\beta-d$ adhuc positivum est, diversum a priori exemplum habetur, ut terminus $\frac{1}{(\beta-2d)(\beta-d)}=\frac{1}{-2.1}=-\frac{1}{2}$ a summatione eximendus et $\gamma=9-4=5$ adhibendum sit. Sic est

$$S=\left\{\begin{array}{l}\frac{1}{4.7}+\frac{1}{10.13}+\ldots\text{ in inf.}\\ +\frac{1}{5.8}+\frac{1}{11.14}+\ldots+\frac{1}{95.98}\\ -\frac{1}{2}\end{array}\right\}$$

Quarum serierum prior methodo secunda optime summatur, qua, quatuor primos terminos modo communi sibi invicem addendo,

$$\begin{aligned}\frac{1}{4.7}&=0{,}0357143\\ \frac{1}{10.13}&=0{,}0076923\\ \frac{1}{16.19}&=0{,}0032895\\ \frac{1}{22.25}&=0{,}0018182\\ &\overline{0{,}0485143}\end{aligned}$$

et expressionis pro serie, quæ superest $\frac{1}{28.31}+\frac{1}{34.37}+\ldots$ in inf. septem terminos computando

$$\begin{array}{r}0{,}0059524\\ 0{,}0002626\\ 0{,}0000394\\ 0{,}0000096\\ 0{,}0000031\\ 0{,}0000012\\ 0{,}0000005\\ \hline 0{,}0062688\end{array}$$

cui addendo 0,0485143

fit series prior 0,0547831 ubi quinta figura decimalis certa est.

Alteram seriem §. 14. invenimus esse = 0,0404750961

cui addendo 0,05478

et subtrahendo 0,5

fit tota series = - 0,40485.